ALGEBRAIC GROUPS: WEEK 8 HOMEWORK

The first two problems below are eligible for submission. The last two are optional exercises.

Let G be a linear algebraic group over an algebraically closed field k.

(1) Let θ be a vector field on G. Show that θ is left-invariant if and only if the diagram on the left commutes, and that θ is right-invariant if and only if the diagram on the right commutes. (Hint: $\ell_g^* = (\mathrm{ev}_g \otimes 1) \circ \Delta$.)

$$\begin{tikzcd} k[G] \arrow[r, "\theta"] \arrow[d, "\Delta"'] & k[G] \arrow[d, "\Delta"] \\ k[G] \otimes k[G] \arrow[r, "1\otimes\theta"] & k[G] \otimes k[G] \end{tikzcd} \qquad \begin{tikzcd} k[G] \arrow[r, "\theta"] \arrow[d, "\Delta"'] & k[G] \arrow[d, "\Delta"] \\ k[G] \otimes k[G] \arrow[r, "\theta\otimes 1"] & k[G] \otimes k[G] \end{tikzcd}$$

(2) Recall that $\mathrm{Lie}(G)_\ell$ denotes the Lie algebra of left-invariant vector fields on G, and $\mathfrak{g} = T_1 G$ denotes the tangent space at the identity of G, with Lie algebra structure coming from $k[G]^*$. We have a map:

$$\begin{aligned} \mathrm{Lie}(G)_\ell &\longrightarrow \mathfrak{g} \\ \theta &\mapsto \eta \circ \theta, \end{aligned}$$

where η is the counit. Prove that this map is an isomorphism of Lie algebras. (Hint: you may find Exercise 1 and the counit axiom useful.)

(3) Suppose G is connected. Show that the Lie algebra of G (in any manifestation) is commutative if and only if G is commutative.

(4) Let $G = \mathbb{G}_a$. Describe the Lie algebra of all vector fields on G. Describe the Lie algebra of left-invariant vector fields on G. Describe the Lie algebra of right-invariant vector fields on G.