ALGEBRAIC GROUPS

Introductory Lecture

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WHAT IS A GROUP?

Reminders:

- A **group** is a set G with an associative operation

$$\mu : G \times G \to G$$

  such that there exists $e \in G$ and a bijection $i : G \to G$ satisfying the following, for all $g \in G$:

$$\mu(e, g) = g = \mu(g, e) \qquad \mu(i(g), g) = e = \mu(g, i(g)).$$

  Examples: symmetric groups, dihedral groups, finite cyclic groups.
- A **Lie group** is a smooth manifold G equipped with the structure of a group, such that the maps μ and i are smooth maps. Example: the general linear group $GL_n(\mathbb{R})$ over the real numbers.

Definition

An **algebraic group** is an algebraic variety equipped with the structure of a group, such that the maps μ and i are morphisms of algebraic varieties.

Main objectives of this class:

- Cover enough algebraic geometry to make sense of this definition.
- Familiarize ourselves with the most important examples of algebraic groups.
- Understand the structure of algebraic groups, especially reductive ones.
- Delve into the representation theory of algebraic groups.

MOTIVATION FOR ALGEBRAIC GROUPS

J. S. Milne: “Without too much exaggeration, one can say that all the theory of algebraic group does is show that the theory of Killing and Cartan for ‘local’ objects over $\mathbb{C}$ extends in a natural way to ‘global’ objects over arbitrary fields.”

In more detail:

- Groups of interest can be defined and understood using the powerful language of algebraic geometry.
- The theory of algebraic groups provides a uniform approach to studying a wide variety of groups over arbitrary fields beyond $\mathbb{R}$ and $\mathbb{C}$.
- Connections to number theory, mathematical physics, etc.

RUNNING EXAMPLE

Let k be an algebraically closed field. Define

$$\mathrm{GL}_2 = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_{2\times 2} \mid ad - bc \neq 0\}.$$

We have some important subgroups of GL_2:

- Diagonal matrices, which form a **maximal torus** of GL_2:

$$T = \{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \mid a, d \in k^\times\}$$

- Upper triangular matrices, which form a **Borel subgroup** of GL_2:

$$B = \{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, d \in k^\times, b \in k\}$$

- The special linear group SL_2 defined via the equation $ad - bc = 1$.

Defining representation: By definition, there is a linear action of GL_2 on the two-dimensional space k^2.

Other representations: A representation of GL_2 is a linear action of GL_2 on a vector space V over k. We can take direct sums, tensor products, symmetric and exterior powers of known representations to get new representations.

Irreducible representations: We have GL_2 can be understood combinatorially via irreducible representations of the center $Z(GL_2) \simeq k^\times$ (which are all one-dimensional) and irreducible representations of SL_2 (which are indexed by $n = 1, 2, 3, \ldots$).

Meta-principle: One understands algebraic groups through their representations. In fact, one can recover a group through information about its representations.

Claim: The quotient GL_2/B is isomorphic to one-dimensional projective space $\mathbb{P}^1$ over k.

Idea: The group GL_2 acts on $\mathbb{P}^1$ via :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot [x : y] = [ax + by : cx + dy]$$

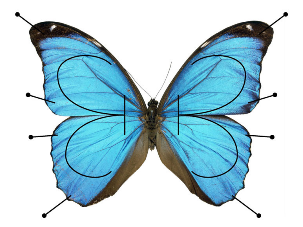
where $[x : y] \in \mathbb{P}^1 = (k^2 \setminus \{0\})/k^\times$. The action[1] is transitive, and the stabilizer of $[1 : 0]$ is B.

Aside: This action coincides with Möbius transformations on the Riemann sphere, and the stabilizer of ∞ is B.

In general: The quotient of a (linear) algebraic group by a Borel subgroup will be a projective variety, known as the flag variety.

[1]Note that this action is **not** a linear action as discussed in the previous slide.

The algebraic groups with the richest structure are the reductive linear algebraic groups.



Following Grothendieck's vision (c.f. Milne):

- The body of the butterfly is a maxmial torus T (e.g. diagonal matrices).
- The wings are opposite Borel subgroups (e.g. upper- and lower-triangular matrices).
- The pins are the root system, which rigidifies this situation.

REST OF TODAY

Algebraic geometry concepts:

- What is the Zariski topology?
- What is an affine algebra and an affine variety?
- What is a variety?
- What is a morphism of varieties?
- What is projective space and a projective variety?