ALGEBRAIC GROUPS: FINAL EXAM

Let k be an algebraically closed field of characteristic 0.

(1) **[15 points]** Recall the usual basis E, F, and H for the Lie algebra $\mathfrak{sl}_2$ of SL_2. Consider the following endomorphisms of $\mathfrak{sl}_2$:

$$\mathrm{Ad}(g), \qquad \mathrm{ad}(E) = [E, -], \qquad \mathrm{ad}(F) = [F, -], \qquad \mathrm{ad}(H) = [H, -],$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2$. Express each of these endomorphisms as a matrix in the basis $\{E, F, H\}$.

(2) **[15 points]** Fix a positive integer n.

(a) Show that the ideal generated by $T^n - 1$ in the polynomial algebra $k[T]$ is radical[1].

(b) Consider the affine variety defined as the Zariski closed set $V(T^n - 1)$ in k. Show that this variety has the natural structure of a linear algebraic group, and identify this group as a subgroup of the multiplicative group $\mathbb{G}_m$.

(c) State the affine algebra of this linear algebraic group, and write out the comultiplication, counit, and anitpode maps.

(3) **[15 points]** Adopt the notation of Exercise 2.1.5 in Springer. Consider the algebra homomorphism $k[x] \to B = k[PSL_2]$ defined by

$$x \mapsto t_1^2 + 2t_1t_4 + t_4^2,$$

and the induced map of varieties $f : PSL_2 \to \mathbb{A}^1$. Describe the subset of semisimple and the subset of unipotent elements of PSL_2 in terms of the fibers of f.

(4) **[15 points]** Let G be a linear algebraic group with Lie algebra $\mathfrak{g} = T_1(G)$. Let X be an affine G-variety with action map $a : G \times X \to X$ and induced algebra homomorphism $a^* : k[X] \to k[G] \otimes k[X]$. As in 2.3.5 of Springer, we have a (generally infinite-dimensional) representation $\rho_X : G \to GL(k[X])$.

[1]Hint: consider the roots of this polynomial.

(a) Prove that the following map is a Lie algebra homomorphism:

$$\begin{aligned} \mathfrak{g} &\longrightarrow \mathcal{T}_X = \mathrm{Der}(k[X], k[X]) \\ \xi &\mapsto -(\xi \otimes 1) \circ a^* \end{aligned}$$

This homomorphism is known as the *infinitesimal action* of G on X.

(b) The inclusion

$$\mathcal{T}_X \hookrightarrow \mathrm{End}(k[X]) = \mathfrak{gl}(k[X])$$

gives $k[X]$ the structure of a representation of the Lie algebra $\mathcal{T}_X$. Precomposing with the infinitesimal action map, we obtain a representation of $\mathfrak{g}$ on $k[X]$. Relate this representation of $\mathfrak{g}$ to the one obtained from the differential of ρ_X.

(5) **[20 points]** Let $k[x, y]$ be the affine algebra of $\mathbb{A}^2$, and let ∂_x and ∂_y denote the partial derivatives as derivations of $k[x, y]$.

(a) Consider the usual action of $G = \mathrm{SL}_2$ on $\mathbb{A}^2$. Compute the infinitesimal action (see Exercise 4)

$$\mu : \mathfrak{sl}_2 \longrightarrow \mathcal{T}_{\mathbb{A}^2}$$

by specifying the images of the standard basis elements E, F, and H in terms of ∂_x and ∂_y. (Observe that it is indeed a homomorphism of Lie algebras.)

(b) For $n \geq 0$, let V_n be the subspace of $k[x, y]$ consisting of homogeneous polynomials of degree n. Prove that each V_n is a representation of $\mathfrak{sl}_2$ via μ, and that it is irreducible.

(c) Identify V_1 with the 'defining' representation of $\mathfrak{sl}_2$ on k^2, and V_2 with the adjoint representation of $\mathfrak{sl}_2$ on itself.

(d) **Optional**. For each $n \geq 0$, define an action of SL_2 on V_n which differentiates to the action of $\mathfrak{sl}_2$ on V_n.

(e) **Optional**. Observe that $k[x, y]$ embeds into $\mathrm{End}(k[x, y])$ as the left multiplication operators. Let $\mathcal{D}_{\mathbb{A}^2}$ be the subalgebra of $\mathrm{End}(k[x, y])$ generated by $\mathcal{T}_{\mathbb{A}^2}$ and the image of $k[x, y]$. This is known as the *algebra of differential operators* on $\mathbb{A}^2$. Argue that μ induces a homomorphism of algebras

$$U\mathfrak{sl}_2 \longrightarrow \mathcal{D}_{\mathbb{A}^2}.$$

Use this homomorphism to compute the action of the Casimir operator $\Delta = EF + FE + \frac{1}{2}H^2$ on $x^n \in V_n$. This gives the *central character* of V_n.

(6) **[20 points]** Let $G = \mathrm{SL}_2$ and B the Borel subgroup of upper-triangular matrices. Fix an integer n.

(a) Define a map $\chi_n : B \to \mathbb{G}_m$ by:

$$\chi_n : \begin{bmatrix} t & u \\ 0 & t^{-1} \end{bmatrix} \mapsto t^n.$$

Verify that χ_n is a character, and show that any character of B is of this form.

(b) Define $\mathcal{L}_n$ as the quotient of $G \times k$ by the equivalence relation

$$(gb, \eta) \sim (g, \chi_n(b)\eta)$$

for all $g \in G$, $b \in B$, and $\eta \in k$. Consider the map

$$p : \mathcal{L}_n \longrightarrow G/B$$
$$[g, z] \mapsto gB$$

Observe that $\mathcal{L}_n$ has a natural action of G by left multiplication making p a G-equivariant map. Show that each fiber of p is a one-dimensional vector space, and the action of $x \in G$ defines a linear isomorphism $p^{-1}(gB) \to p^{-1}(xgB)$. Hence $\mathcal{L}_n$ is *G-equivariant line bundle* on G/B.

(c) A section of $\mathcal{L}_n$ is map of varieties $s : G/B \to \mathcal{L}_n$ such that $p \circ s$ is the identity on G/B. The space of sections is denoted $\Gamma(G/B, \mathcal{L}_n)$. Verify that $\Gamma(G/B, \mathcal{L}_n)$ is a vector space, and define a linear action of G on $\Gamma(G/B, \mathcal{L}_n)$.

(d) **Optional**. For $n \leq 0$, define an isomorphism of SL_2-representations between the $\Gamma(\mathbb{P}^1, \mathcal{L}_n)$ and space V_{-n} from Exercise 5.