UNIPOTENT AND COMMUTATIVE GROUPS

1. Unipotent groups

Definition 1.1. For $n \geq 1$, let U_n denote the subgroup of GL_n consisting of upper-triangular unipotent matrices.

Proposition 1.2. *Let G be a unipotent linear algebraic group. The only finite-dimensional irreducible representation of G is the trivial one.*

Idea of proof. Let $\rho : G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G. As in the proof of 2.4.12, one uses Burnside's theorem on matrix algebras and the non-degeneracy of the trace pairing to argue that the image $\rho(G)$ in $\mathrm{GL}(V)$ is trivial. Since V is an irreducible representation of G, it follows that V is the trivial one-dimensional representation. □

Remark 1.3. A unipotent group may have non-trivial finite-dimensional indecomposable representations. For example, for $n \geq 2$, the natural action of U_n on k^n is indecomposable.

Proposition 1.4. *Let G be a subgroup of GL_n consisting of unipotent matrices. There exists $x \in \mathrm{GL}_n$ such that xGx^{-1} is contained in U_n.*

Idea of proof. Proceed by induction on n. The case $n = 1$ is immediate since the only unipotent element of $\mathrm{GL}_1 = \mathbb{G}_m$ is 1. Suppose $n > 1$. Consider the action of G on $V = k^n$. By Proposition 1.2, this action must be reducible, i.e. there is a proper non-zero G-stable subspace W of V. Apply induction to the action of G on W and V/W (details omitted). □

2. Commutative groups

Proposition 2.1. *Let G be a connected linear algebraic group of dimension 1. Then G is commutative.*

We need a bit of set-up before proving this proposition.

Definition 2.2. Let G be an algebraic group. For $g \in G$, we denote the conjugacy class of g by $C_g := \{xgx^{-1} \ : \ x \in G\}$.

Lemma 2.3. *Let G be a connected linear algebraic group of dimension 0 or 1. If the closure $\overline{C_g}$ is equal to G for some $g \in G$, then G is the trivial group.*

Idea of proof. Suppose $\overline{C_g} = G$ for some $g \in G$. One proceeds in a similar fashion to the proof of 3.1.3. The steps are as follows:

(1) The facts that $\overline{C_g} = G$ and $\dim(G) \leq 1$ together imply that the complement $G \setminus C_g$ is finite (and possibly empty).
(2) Embed G into GL_n. Since the characteristic polynomial is conjugation invariant, step 1 implies the restriction of the characteristic polynomial map on GL_n to G has finite image. Moreover, since G is connected and the characteristic polynomial map is continuous, every element of G has the same characteristic polynomial as $1 \in G$.
(3) It follows that every element of G is unipotent, and we can assume (Proposition 1.4) that the image of G lies in U_n. Since the derived series of U_n is eventually zero, the same is true for G. In particular, the commutator $[G, G]$ must be a proper subgroup of G. Since $\dim(G) \leq 1$ and $[G, G]$ is connected, the only possibility is that $[G, G]$ is trivial.
(4) Finally, observe that

$$\{g\} \subseteq C_g = g^{-1}gC_g \subseteq g[G, G] = g\{e\} = \{g\}.$$

Hence $C_g = g$ is a single point. Since we also have $\overline{C_g} = G$, we conclude that G is the trivial group.

□

Proof of Proposition 2.1. Let G be a connected linear algebraic group of dimension 1 and let $g \in G$. We argue that $C_g = \{g\}$. Since G is not trivial, Lemma 2.3 implies that the closure $\overline{C_g}$ is proper in G. Moreover, $\overline{C_g}$ is an irreducible closed subset of G, and G is connected of dimension one, so we must have that $\overline{C_g}$ is of dimension zero. Since it is also connected, it follows that $\overline{C_g} = \{g\}$ for all $g \in G$. Hence $C_g = \{g\}$, as desired. □

Theorem 2.4. *Let G be a linear algebraic group of dimension 1. Then G is isomorphic either to $\mathbb{G}_m$ or $\mathbb{G}_a$.*

If G is a linear algebraic group of dimension 1, we can deduce from earlier results that either $G = G_s$ or $G = G_u$. We will prove later that G is isomorphic to $\mathbb{G}_m$ in the former case. In the latter, case G is isomorphic to $\mathbb{G}_a$ in the latter case, but we will not give the full proof in this class (see Sections 3.3 and 3.4).