Torus-valued moment maps

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In this document we state the definition of a torus-valued moment map, prove basic properties of these maps, and give an example.

1 Review of ordinary moment maps

Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Suppose G acts on a smooth symplectic variety (X, ω) preserving the symplectic form.

Definition 1.1. For $\xi \in \mathfrak{g}$, define a vector field v_ξ on X by $(v_\xi)_x = d(a_x)_e(\xi)$ where $a_x : G \to X$ is the map taking g to gx. We say that v_ξ is the vector field generated by the infinitesimal action of ξ.

Let $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ denote a choice of an invariant positive definite inner product on $\mathfrak{g}$ which we use to identify $\mathfrak{g} \simeq \mathfrak{g}^*$.

Definition 1.2. A moment map for the action of G on X is a smooth map $\phi : X \to \mathfrak{g}$ that satisfies

$$\omega(v_\xi, -) = \langle d\phi, \xi \rangle$$

for any $\xi \in \mathfrak{g}$.

The equation can be written locally as $\omega_x((v_\xi)_x, v) = \langle (d\phi)_x(v), \xi \rangle$, for $x \in X$ and $v \in T_x X$, where we identify the tangent space $T_\xi \mathfrak{g}$ with $\mathfrak{g}$ itself.

2 Definition of torus-valued moment maps

Let $L_{g^{-1}} : G \to G$ denote the left translation action of g^{-1} taking h to $g^{-1}h$.

Definition 2.1. The left-invariant Maurer-Cartan form $\theta \in \Omega^1(G, \mathfrak{g})$ is defined as

$$\theta_g = d(L_{g^{-1}})_g : T_g G \to \mathfrak{g} = T_e G.$$

The notion of a group-valued moment map $X \to G$ is introduced in [AMM]. Here we recall the definition in the case that $G = T$ is a complex algebraic torus, and X is a complex manifold. Throughout, we consider the adjoint (equivalently, trivial) action of T on itself.

Definition 2.2. A torus-valued moment map is a smooth, T-equivariant map $\mu : X \to T$ such that

$$\omega(v_\xi, -) = \langle \mu^* \theta, \xi \rangle.$$

The corresponding homomorphism $\mu^\# : \mathcal{O}(T) \to \mathcal{O}(X)$ is called a *comoment map*.

To be explicit, the condition can be expressed locally as $\omega_x((v_\xi)_x, v) = \langle \theta_{\mu(x)}(d\mu_x(v)), \xi \rangle$, for $x \in X$ and $v \in T_x X$.

Proposition 2.3. *Choose an isomorphism $T \simeq (\mathbb{C}^\times)^n$. A T-equivariant smooth map $\mu : X \to T$ is a group-valued moment map if and only if*

$$\omega(v_\xi, -) = \sum_{i=1}^n \frac{d\mu_i}{\mu_i}\xi_i,$$

where μ_i is the composition of μ with the ith projection.

Proof. The Maurer-Cartan form for T can be written as $[\theta_g(v)]_i = g_i^{-1}v_i$ for $g \in T \simeq (\mathbb{C}^\times)^n$ and $v \in T_gT \simeq \mathbb{C}^n$, and the pairing $\langle , \rangle$ can be taken to be the usual dot product. □

Exercise: Check that the moment map condition is compatible with the the skew-symmetry of ω.

Remark 2.4. We see from the reformulation above that a map $\mu : X \to T$ is a torus-valued moment map if and only if a branch of $\log \circ \mu$ is locally an ordinary moment map for the action of T on X.

3 Properties of torus-valued moment maps.

Let $K = (\mathbb{C}^\times)^d$ and $T = (\mathbb{C}^\times)^n$ be standard tori of rank d and n where $d < n$. Let $\phi : K \hookrightarrow T$ be an inclusion. Thus, ϕ has the form $\phi(k)_i = \prod_{j=1}^d k_j^{m_{ij}}$, $i = 1, \dots, n$, for some integers $m_{ij} \in \mathbb{Z}$. There is a 'transpose' map $\phi^\dagger : T \to K$ defined by $\phi^\dagger(t)_j = \prod_{i=1}^n t_i^{m_{ij}}$. Let $H = T/K$ be the quotient torus. Denote by $\psi : T \to H$ the quotient map, and $\psi^\dagger : H \to K$ its transpose.

Proposition 3.1. *If $\mu : X \to T$ is a moment map, then the composition $\phi^\dagger \circ \mu$ is a moment map for the action of K on X induced by ϕ.*

First proof. Let $\mathfrak{k}$ denote the Lie algebra of K and let $\mathrm{Lie}(\phi) : \mathfrak{k} \to \mathfrak{t}$, and $\mathrm{Lie}(\phi^\dagger) = \mathrm{Lie}(\phi)^T : \mathfrak{t} \to \mathfrak{k}$ denote the induced Lie algebra homomorphisms. We use the same notation for the maps on 1-forms:

$$\mathrm{Lie}(\phi) : \Omega^1(-, \mathfrak{k}) \to \Omega^1(-, \mathfrak{t}), \qquad \mathrm{Lie}(\phi^\dagger) : \Omega^1(-, \mathfrak{t}) \to \Omega^1(-, \mathfrak{k}).$$

These maps commute with the pullback of 1-forms along smooth maps. Let θ_T and θ_K be the Maurer-Cartan forms on T and K. Then[1] $(\phi^\dagger)^*\theta_K = \mathrm{Lie}(\phi^\dagger)(\theta_T)$.

For $\zeta \in \mathfrak{k}$, write v_ζ^K for the vector fields on X generated by ζ. In fact, v_ζ^K coincides with the vector field $v^T_{\mathrm{Lie}(\zeta)}$ generated by the image of ζ in $\mathfrak{t}$. The remainder of the proof is a computation:

$$\begin{aligned}\langle(\phi^\dagger \circ \mu)^*\theta_K, \zeta\rangle &= \langle \mu^* \circ (\phi^\dagger)^*\theta_K, \zeta\rangle = \langle \mu^*(\mathrm{Lie}(\phi^\dagger)(\theta_T)), \zeta\rangle = \langle \mathrm{Lie}(\phi^\dagger)(\mu^*\theta_T), \zeta\rangle \\ &= \langle \mu^*\theta_T, \mathrm{Lie}(\phi)(\zeta)\rangle = \omega(v^T_{\mathrm{Lie}(\zeta)}, -) = \omega(v^K_\zeta, -).\end{aligned}$$

□

Second proof. Adopt the notation of the previous proof. Observe that $(\phi_*\zeta)_i = \sum_{j=1}^d m_{ij}\zeta_j$. We have:

$$\omega(v^K_\zeta, -) = \omega(v^T_{\mathrm{Lie}(\phi)\zeta}, -) = \sum_{i=1}^n \frac{d\mu_i}{\mu_i}(\mathrm{Lie}(\phi)(\zeta))_i = \sum_{i=1}^n\sum_{j=1}^d \frac{m_{ij}\zeta_j d\mu_i}{\mu_i}.$$

[1] In fact, for any group homomorphism $\beta : G_1 \to G_2$, it is easy to show that $\beta^*\theta_{G_2} = \mathrm{Lie}(\beta)(\theta_{G_1})$.

On the other hand, let $(\phi^\dagger \circ \mu)_j$ denote the composition of $\phi^\dagger \circ \mu$ with the jth projection, so $(\phi^\dagger \circ \mu)_j(x) = \prod_{i=1}^n \mu_i(x)^{m_{ij}}$. Then one computes:

$$d[(\phi^\dagger \circ \mu)_j] = \sum_{i=1}^n m_{ij}\mu_i^{m_{ij}-1} d\mu_i \prod_{i'=1,i'\neq i}^n \mu_{i'}^{m_{i'j}} = \sum_{i=1}^n \frac{m_{ij}(\phi^\dagger \circ \mu)_j d\mu_i}{\mu_i}.$$

Therefore,

$$\sum_{j=1}^d \frac{d[(\phi^\dagger \circ \mu)_j]}{(\phi^\dagger \circ \mu)_j}\zeta_j = \sum_{i=1}^n \sum_{j=1}^d \frac{m_{ij}\zeta_j d\mu_i}{\mu_i} = \omega(v_\zeta^K, -).$$

The claim follows. □

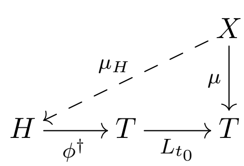
Remark 3.2. In the proposition and its proof, we did not use the hypothesis that ϕ is injective. The same result, with the same proof, applies to an arbitrary homomorphism $\phi : T \to K$ of tori.

Lemma 3.3. *Suppose the action of T on X is trivial. Then $\mu : X \to T$ is moment map if and only if μ is a constant map.*

Proof. In this case, the vector field v_ξ is the zero for any $\xi \in \mathfrak{t}$. Thus, μ is a moment map if and only if $\mu^*\theta = 0$. This is equivalent to $\mu_i(x)^{-1}(d(\mu_i)_x(v)) = 0$ for all $x \in X$, $v \in T_xX$, $i = 1, \ldots, n$. Since $\mu_i(x)$ is nonzero, the above holds if and only if $d(\mu_i)_x = 0$ for all $x \in X$, $i = 1, \ldots, n$, i.e. if and only if μ_i is constant for all $i = 1, \ldots, n$. The lemma follows. □

Lemma 3.4. *Suppose $\mu : X \to T$ is a moment map, and the action of K on X is trivial. Then there is an induced action of H on X and a moment map $\mu_H : X \to H$ that satisfies $\mu = \mu_H \circ \phi^\dagger \circ L_{t_0}$ for some $t_0 \in T$.*

In other words, the following diagram commutes:



Proof. Let $\mathfrak{h}$ denote the Lie algebra of H. As in the first proof of Proposition 3.1, we have Lie algebra homomorphisms $\mathrm{Lie}(\psi) : \mathfrak{t} \to \mathfrak{h}$ and $\mathrm{Lie}(\psi^\dagger) = \mathrm{Lie}(\psi)^T : \mathfrak{h} \to \mathfrak{t}$. The short exact sequence of Lie algebras $0 \to \mathfrak{h} \to \mathfrak{t} \to \mathfrak{k} \to 0$, with maps $\mathrm{Lie}(\psi^\dagger)$ and $\mathrm{Lie}(\phi^\dagger)$, exponentiates to a short exact sequence

$$1 \longrightarrow H \xrightarrow{\psi^\dagger} T \xrightarrow{\phi^\dagger} K \longrightarrow 1.$$

Fix $x_0 \in X$ and let $t_0 = \mu(x_0) \in T$. Since the action of K on X is trivial and $\phi^\dagger \circ \mu : X \to K$ is a moment map, the preceeding lemma implies that $\phi^\dagger \circ \mu$ is constant. Thus, $\mu(x)t_0^{-1} \in \mathrm{Ker}(\phi^\dagger) = \mathrm{Im}(\psi^\dagger)$ for any $x \in X$. Using the fact that $\phi^\dagger$ is injective, define

$$\begin{aligned} \mu_H &: X \to H \\ & x \mapsto (\phi^\dagger)^{-1}\left(\frac{\mu(x)}{t_0}\right) \end{aligned}$$

We show that μ_H is a moment map. Let $\xi \in \mathfrak{t}$. The vector field $v^H_{\mathrm{Lie}(\psi)(\xi)}$ corresponding to the image of ξ in $\mathfrak{h}$ coincides with the vector field v^T_ξ corresponding to ξ. Since $\mathrm{Lie}(\psi)$ is surjective, the result is a consequence of the following computation, which uses facts stated in the first proof of Proposition 3.1, and the left-invariance of θ_T:

$$\begin{aligned}\omega(v^H_{\mathrm{Lie}(\psi)(\xi)}, -) = \omega(v^T_\xi, -) = \langle \mu^*\theta_T, \xi\rangle = \langle \mu_H^* \circ (\psi^\dagger)^* \circ L_{t_0}^*\theta_T, \xi\rangle = \langle \mu_H^* \circ (\psi^\dagger)^*\theta_T, \xi\rangle \\ = \langle \mu_H^*(\mathrm{Lie}(\psi^\dagger)(\theta_H)), \xi\rangle = \langle \mathrm{Lie}(\phi^\dagger)(\mu_H^*\theta_H), \xi\rangle = \langle \mu_H^*\theta_H, \mathrm{Lie}(\phi)(\xi)\rangle.\end{aligned}$$

□

4 Example

There is an action of $T = (\mathbb{C}^\times)^n$ on the cotangent bundle $T^*\mathbb{C}^n$ by componentwise scaling: $(t \cdot (p,\omega))_i = (t_i p_i, t_i^{-1}\omega_i)$. Precomposition by ϕ induces an action of K on $T^*\mathbb{C}^n$. Fix the following notation:

$$\mathcal{O}(T^*\mathbb{C}^n) = \mathbb{C}[x_i, \partial_i] = \mathbb{C}[x_i, \partial_i \mid i = 1, \ldots, n], \qquad \mathcal{O}(T^*\mathbb{C}^n)^\circ = \mathbb{C}[x_i, \partial_i][(1 + x_i\partial_i)^{-1}].$$

$$(T^*\mathbb{C}^n)^\circ = \{(p, w) \in T^*\mathbb{C}^n \ : \ 1 + p_i w_i \neq 0\}.$$

Equip $(T^*\mathbb{C}^n)^\circ$ with the symplectic form $\omega = \sum_i \frac{dp_i \wedge dw_i}{1+p_i w_i}$.

Proposition 4.1. *The following are group-valued moment maps:*

$$\begin{aligned} \mu_T : (T^*\mathbb{C}^n)^\circ &\to T & \qquad \mu_K : (T^*\mathbb{C}^n)^\circ &\to K \\ (p, w) &\mapsto 1 + p_i w_i & (p, w) &\mapsto (\prod_{i=1}^n (1 + p_i w_i)^{m_{ij}})_j. \end{aligned}$$

Proof. For $(p, w) \in (T^*\mathbb{C}^n)^\circ$, we write $\{\partial p_i, \partial w_i \mid i = 1, \ldots, n\}$ and $\{dp_i, dw_i \mid i = 1, \ldots, n\}$ for the natural bases of $2n$-dimensional vector spaces $T_{(p,w)}T^*\mathbb{C}^n$ and $T^*_{(p,w)}T^*\mathbb{C}^n$. For $\xi \in \mathfrak{t} = \mathbb{C}^n$ we have $(v_\xi)_{(p,w)} = \sum_{i=1}^n \xi_i p_i(\partial p_i) - \xi_i w_i(\partial w_i)$. Therefore, for any $i = 1, \ldots, n$,

$$\omega_{(p,w)}((v_\xi)_{(p,w)}, \partial p_i) = \omega_{(p,w)}(\sum_{i'=1}^n \xi p_{i'}\partial p_{i'} - \xi w_{i'}\partial w_{i'}, \partial p_i) = \frac{dp_i \wedge dw_i}{1 + p_i w_i}(-\xi w_i \partial w_i, \partial p_i) = \frac{\xi w_i}{1 + p_i w_i}.$$

Similarly,

$$\omega_{(p,w)}((v_\xi)_{(p,w)}, \partial w_i) = \frac{\xi p_i}{1 + p_i w_i}.$$

Therefore,

$$\omega_{(p,w)}((v_\xi)_{(p,w)}, -) = \sum_{i=1}^n \frac{\xi_i w_i(dp_i) + \xi_i p_i(dw_i)}{1 + p_i w_i}.$$

On the other hand, $\frac{(d\mu_i)_{(p,w)}}{\mu_i(p,w)} = \frac{w_i dp_i + p_i dw_i}{1+p_i w_i}$. The claim for $\mu = \mu_T$ now follows, and the claim for μ_K is a consequence of Proposition 3.1. □

References

[AMM] A. Alekseev, A. Malkin, E. Meinrenken. Lie group valued moment maps, *Journal of Differential Geometry* 48 (1998), 445–495.