The cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and related topics

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Abstract

The subgroup U of unit quaternions is isomorphic to SU(2) and is a double cover of SO(3). This allows a simple computation of the fundamental group of SO(n). We also show how $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a double cover of SO(4). Finally, we argue that O(4) is generated by the left and right multiplication maps together with quaternionic conjugation.

1 A brief review of the quaternions $\mathbb{H}$

Let $\mathbb{H}$ be the free $\mathbb{R}$-module on the set $\{1, i, j, k\}$. Therefore, $\mathbb{H}$ is a four-dimensional vector space in which an arbitrary element x can be written as $x = a + bi + cj + dk$ for some real numbers $a, b, c,$ and d. Define an algebra structure on $\mathbb{H}$ by extending linearly the multiplication of the finite group of quaternions

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

By a slight abuse of terminology, elements of $\mathbb{H}$ are called *quaternions*. The conjugate of an element $x = a + bi + cj + dk$ of $\mathbb{H}$ is defined as $\bar{x} = a - bi - cj - dk$, and the map $x \mapsto x\bar{x}$ defines a norm on $\mathbb{H}$. The subspace $\mathrm{Im}\mathbb{H}$ of purely imaginary quaternions is the subspace with $a = 0$. The unit sphere in $\mathbb{H}$, i.e. the *unit quaternions*, forms a multiplicative group, denoted U, that is diffeomorphic to S^3.

Lemma 1. *The center of* $\mathbb{H}$ *is* $\mathrm{Span}\{1\}$.

Proof. It is clear that $\mathrm{Span}\{1\}$ is contained in the center since the center of Q_8 is $\{\pm 1\}$. Suppose $x = a + bi + cj + dk$ is in the center. Then $xi = ix$ implies that

$$ai - b - ck + dj = ai - b + ck - dj,$$

so $c = d = 0$ and $x = a + bi$. Now, $xj = jx$ implies that

$$aj + bk = aj - bk.$$

Hence $b = 0$ and $x = a \in \mathrm{Span}\{1\}$. □

2 $U \simeq \mathbf{SU}(2)$

We argue that, as Lie groups, U is isomorphic to the special unitary group SU(2). For each $g \in U$, there is a map

$$\begin{aligned} H \simeq \mathbb{R}^4 &\to H \simeq \mathbb{R}^4 \\ x &\mapsto gx. \end{aligned}$$

This map is $\mathbb{R}$-linear since

- the elements of Span$\{1\}$, which are the scalars in this case, are in the center of $\mathbb{H}$, and
- multiplication distributes over addition in the ring $\mathbb{H}$.

Moreover, the map is an isometry by the multiplicativity of the norm: $|gx| = |g||x| = |x|$ for any $g \in U$ and $x \in H$. Hence U acts by isometries on $\mathbb{R}^4$ by left multiplication. An identical argument confirms that right multiplication defines an action by isometries of U on $\mathbb{R}^4$.

Let $R_i : \mathbb{H} \to \mathbb{H}$ denote right multiplication by i. Then $(R_i)^2 = -\operatorname{Id}$. Define a nondegenerate symmetric bilinear form on $\mathbb{H}$ as

$$B(a + bi + cj + dk, x + yi + zj + wk) = ax + by + cz + dw.$$

The norm defined above on $\mathbb{H}$ is the same as the norm induced by B. A short calculation shows that $B(R_i(v), w) = -B(v, R_i(w))$ for any $v, w \in \mathbb{H}$. The maps R_i and B are enough to define a complex vector space structure with a Hermitian form on $\mathbb{H}$. Specifically, $a+bi \in \mathbb{C}$ acts as

$$(a + bi) \cdot v = a \cdot v + b \cdot (R_i(v)) = av + b(iv),$$

and the Hermitian form H is defined as

$$H(v, w) = B(v, w) + i \cdot \omega(v, w)$$

where $\omega(v, w) := -B(R_i(v), w)$ is a skew-symmetric nondegenerate bilinear form (called a *symplectic form*). In particular, $\omega(v, v) = 0$ for all $v \in \mathbb{H}$, and the norm induced by H is equal to the norm on $\mathbb{H}$ defined above:

$$|v|_H = H(v, v) = B(v, v) = a^2 + b^2 + c^2 + d^2 = |v|.$$

Since $\mathbb{H}$ is a 2-dimensional complex vector space, the group SU(2) can be identified with the norm-preserving transformations of $\mathbb{H}$. Each element g of U defines a such transformation by the left multiplication map L_g. Identifying L_g with g, we see that U embeds as a subgroup in SU(2). Both are connected 3-dimensional Lie groups, so U is isomorphic to SU(2).

Here's another way to see the isomorphism. Recall that SU(2) is the set of 2 by 2 matrices A over $\mathbb{C}$ with $\bar{A}^T A = I$ with determinant 1. A standard argument shows the first equality below, and the rest follow:

$$\begin{aligned}
\mathrm{SU}(2) &= \{\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \ : \ a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1\} \\
&= \{\begin{bmatrix} x+iy & u+iv \\ -u+iv & x-iy \end{bmatrix} \ : \ x, y, u, v \in \mathbb{R}, \ x^2+y^2+u^2+v^2 = 1\} \\
&= \{x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + u\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + v\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \ : \\
&\qquad\qquad x, y, u, v \in \mathbb{R}, \quad x^2+y^2+u^2+v^2 = 1\}
\end{aligned}$$

A generic element $x = a + bi + cj + dk$ of $\mathbb{H}$ can be written as $x = (a + bi) + (c + di)j$. This produces a decomposition $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j \simeq \mathbb{C}^2$. Since $j^2 = -1$, this defines a complex structure on $\mathbb{H}$. With this decomposition in mind, the elements 1,i, j, and k act on the right as

$$1 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad i : \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad j : \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad k : \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

These identifications reveal the isomorphism $U \simeq \mathrm{SU}(2)$.

3 The cover $\mathbf{SU}(2) \to \mathbf{SO}(3)$

We will make use of the following lemma, which is proven assuming results from *Introduction to Smooth Manifolds* by John Lee.

Lemma 2. *Suppose $f : G \to H$ is a map of Lie groups of the same dimension, with H connected. If the kernel of f is discrete, then f is a covering map.*

Proof. As a Lie group homomorphism, f has constant rank. The rank is equal to the codimension of the kernel, which in this case is $\dim G - 0 = \dim G = \dim H$. Hence f has full rank and is in particular a local diffeomorphism. Therefore, a neighborhood of the identity in G maps diffeomorphically to a neighborhood of the identity of H. The connectivity of H implies that H is generated by any open neighborhood of the identity. Consequently, f is surjective. Since the kernel is discrete, f must be a covering map. □

Recall that an $\mathbb{R}$-algebra endomorphism of $\mathbb{H}$ is a ring homomorphism from $\mathbb{H}$ to itself that fixes 1. Let $\mathrm{Aut}(\mathbb{H})$ denote the invertible $\mathbb{R}$-algebra endomorphisms of $\mathbb{H}$. Then $\mathrm{Aut}(\mathbb{H})$ is a closed subgroup of $\mathrm{GL}(\mathbb{H}) \simeq \mathrm{GL}_4\mathbb{R}$, hence a Lie group. There is a Lie group homomorphism

$$\begin{aligned} H - \{0\} &\to \mathrm{Aut}(\mathbb{H}) \\ g &\mapsto (x \mapsto gxg^{-1}) \end{aligned}$$

whose kernel is the center $Z(\mathbb{H})$ of $\mathbb{H}$. Observe that each g acts by isometries since $|gxg^{-1}| = |g||x||g^{-1}| = |gg^{-1}||x| = |x|$ for any $x, g \in H$. Each g fixes 1, hence fixes

$$1^\perp = \{x \in \mathbb{H} : B(1, x)\} = \{a + bi + cj + dk \in \mathbb{H} : a = 0\} = \mathrm{Im}\,\mathbb{H}$$

as well. Thus each g acts my isometries fixing 0 on $\operatorname{Im}\mathbb{H}$. Identifying $\operatorname{Im}\mathbb{H}$ with $\mathbb{R}^3$, we see that the homomorphism above induces a map

$$H - \{0\} \to \mathrm{O}(3).$$

The codomain can be refined to special orthogonal group SO(3) since $H - \{0\}$ is connected. We restrict this map to U to obtain a Lie group homomorphism

$$\phi : U \to \mathrm{SO}(3)$$

whose kernel is the discrete subgroup $Z(\mathbb{H}) \cap U = \{\pm 1\}$. Using the isomorphism of the previous section and Lemma 2, it follows that $\phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a 2-fold covering map. Note that SU(2) is simply connected since $\mathrm{SU}(2) \simeq U$ is diffeomorphic to S^3. The 2-fold cover ϕ implies that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$. In fact, SO(3) is diffeomorphic to $\mathbb{RP}^3$.

4 The fundamental group of SO(n)

Th orthogonal group SO(2) consists of rotations about the origin in $\mathbb{R}^2$; it is therefore isomorphic to the circle S^1 and its fundamental group is $\mathbb{Z}$. For any n, SO(n) acts transitively on S^{n-1} with stabilizer SO($n-1$). Hence, there is a fibration

$$\begin{array}{ccc} \mathrm{SO}(n-1) & \hookrightarrow & \mathrm{SO}(n) \\ & & \downarrow \\ & & S^{n-1} \end{array}$$

with associated long exact sequence in homotopy given by

$$\cdots \to \pi_2(S^{n-1}) \to \pi_1(\mathrm{SO}(n-1)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(S^{n-1}) \to 1.$$

For $n \geq 4$, the homotopy group $\pi_2(S^{n-1})$ is trivial, so we obtain an isomorphism

$$\pi_1(\mathrm{SO}(n-1)) \simeq \pi_1(\mathrm{SO}(n)).$$

Since $\mathrm{SO}(3) = \mathbb{Z}/2$, by induction we conclude that

$$\pi_1(\mathrm{SO}(n)) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}/2 & \text{if } n \geq 3 \end{cases}.$$

5 The cover $\mathbf{SU}(2) \times \mathbf{SU}(2) \to \mathbf{SO}(4)$

For each pair $(g, h) \in U \times U$, there is a map on $\mathbb{H} \simeq \mathbb{R}^4$ defined by $x \mapsto gxh$. An argument similar to that in part (c) shows that each such map is $\mathbb{R}$-linear and norm-preserving, hence an element of SO(4). Since $U \times U$ is connected, we obtain a map

$$\begin{aligned} \phi : U \times U &\to \mathrm{SO}(4) \\ (g, h) &\mapsto (x \mapsto gxh). \end{aligned}$$

It is easy to check that this is a Lie group homomorphism. We will show that the kernel of ϕ is the discrete subgroup $\{(1,1),(-1,-1)\}$. By Lemma 2, it will follow that ϕ is a 2-fold covering map.

If $gxh = x$ for all $x \in \mathbb{H}$, then, in particular, $gh = g1h = 1$, so $h = g^{-1}$. From previous work, we know that $x = gxg^{-1}$ for all $x \in \mathbb{H}$ if and only if $g \in \{\pm 1\}$. Hence either $g = 1$ and $h = g^{-1} = 1$ or $g = -1$ and $h = g^{-1} = -1$.

6 Generators for $\mathrm{O}(4)$

The last result we show is that right and left multiplication together with quaternionic conjugation generate $\mathrm{O}(4)$. The quaternionic conjugation map $c : x \mapsto \bar{x}$ is an orientation-reversing orthogonal transformation of $\mathbb{H} \simeq \mathbb{R}^4$ since it involves 3 reflections. Hence c is not in $\mathrm{SO}(4)$, but the other component of $\mathrm{O}(4)$. Let $L_c : \mathrm{O}(4) \to \mathrm{O}(4)$ be left multiplication by c, so $L_c(f) = c \circ f$. Then, by standard Lie group arguments, L_c is a diffeomorphism taking the connected component of the identity diffeomorphically to the connected component of c. In this case, L_c takes $\mathrm{SO}(4)$ diffeomorphically to the component $\mathrm{O}(4) \setminus \mathrm{SO}(4)$. Since the map ϕ from above is a covering map of $\mathrm{SO}(4)$, we observe that

$$L_c \circ \phi : U \times U \to \mathrm{O}(4) \setminus \mathrm{SO}(4)$$

is also a covering map. To summarize, the map

$$\psi : U \times U \times \mathbb{Z}/2 \to \mathrm{O}(4)$$

defined by

$$(g, h, i) \mapsto (x \mapsto c^i(gxh))$$

is a covering map of $\mathrm{O}(4)$. Hence right and left multiplication together with quaternionic conjugation generate $\mathrm{O}(4)$.