Notes on Quantum Hamiltonian Reduction

Iordan Ganev

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This informal document collects some facts on quantum Hamiltonian reduction. The first section establishes notation and gives reminders on coalgebras and Hopf algebras. In the second section, the notion of a quantum moment map is used to perform a Hamiltonian reduction procedure. The ground field is $\mathbb{C}$.

1 Preliminaries on coalgebras, bialgebras, and Hopf algebras

Let A be an algebra with multiplication map $m : A \otimes A \to A$. The associativity axiom implies that, for any positive integer n, there is a single, unambiguous n-fold multiplication map $m^{(n)} : A^{\otimes n} \to A$. In other words, any way of associating a string of n elements of A gives rise to the same product.

Similarly, if (H, Δ, ϵ) is a coalgebra, then the coassociativity axiom implies that, for any positive integer n, there is a single, unambiguous n-fold comultiplication map $\Delta^{(n)} : H \to H^{\otimes n}$. In (sumless) Sweedler notation, the this map is written as

$$\Delta^{(n)}(h) = h_{(1)} \otimes h_{(2)} \otimes \cdots \otimes h_{(n)}.$$

Recall that the symbol $h_{(i)}$ has no intrinsic meaning; it is part of an implicit sum and the values in that sum depend on n. To facilitate clarity in subsequent proofs, we add expressions of the form "$(n = 4)$" in lines involving Sweedler notation. The counit axiom implies that, for $1 \leq i \leq n+1$, the following identity holds:

$$h_{(1)} \otimes \cdot \otimes \epsilon(h_{(i)}) \otimes \cdots \otimes h_{(n+1)} = h_{(1)} \otimes \cdots \otimes h_{(n)}, \tag{1.1}$$

or, equivalently, $(1 \otimes \cdots \otimes \epsilon \otimes \cdots \otimes 1) \circ \Delta^{(n+1)} = \Delta^{(n)}$, where ϵ is in any of the $n+1$ slots.

Now suppose that H is a bialgebra. The comultiplication and counit give the abelian category H-mod of modules for H the sturcture of a tensor category. Explicitly, if M and N are two H-modules, the tensor product of vector spaces $M \otimes N$ has an action of H given by $h(m \otimes n) = h_{(1)}m \otimes h_{(2)}n$. The unit $\mathbb{1}$ of H-mod is the one-dimensional vector space $\mathbb{C}$ is endowed with the H-module structure via the counit map ϵ. The space of invariants M^H of an H-module M is defined as

$$M^H = \mathrm{Hom}_H(\mathbb{1}, M) = \{m \in M \mid h \cdot m = \epsilon(h)m \ \text{ for all } h \in H\}.$$

Observe that functor of taking invariants $(-)^H : H\text{-mod} \to \mathrm{Vec}_{\mathbb{C}}$ is equivalent to the functor $\mathrm{Hom}_H(\mathbb{1}, -)$ and hence is left exact.

Finally, suppose H is a Hopf algebra with antipode S. The **adjoint action** of H on itself is given by

$$h \triangleleft h' = h_{(1)} h' S(h_{(2)}).$$

This formula defines an action since the comultiplication map Δ is an algebra homomorphism and the anitpode S is an algebra antihomomorphism.

2 Quantum moment maps and Hamiltonian reduction

Definition 2.1. A **quantum moment map** is an algebra homomorphism $\mu : H \to A$ from a Hopf algebra H to an algebra A.

For example, one should think of H as the enveloping algebra $\mathcal{U}(\mathfrak{g})$ or quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a Lie algebra, or as the Hopf algebra $\mathcal{O}(G)$ of functions on an algebraic group G. A quantum moment map induces an adjoint action of H on A given by the the formula

$$h \triangleleft a = \mu(h_{(1)}) \cdot a \cdot \mu(S(h_{(2)})),$$

where S is the antipode on H and the multiplication occurs in A. Since μ is an algebra homomorphism, it is immediate that μ is H-equivariant, where H acts on itself by the adjoint action.

Lemma 2.2. *Endowed with the adjoint action, A is an algebra object in the tensor category H-mod.*

Proof. We must show that the multiplication map $m : A \otimes A \to A$ is a map of H-modules. Recall that H acts on the tensor product $A \otimes A$ via the comultiplication map Δ. For any $a \otimes b \in A \otimes A$, we have

$$\begin{aligned}
m(h \triangleleft (a \otimes b)) &= m((h_{(1)} \triangleleft a) \otimes (h_{(2)} \triangleleft b)) && (n = 2)\\
&= \mu(h_{(1)}) \cdot a \cdot \mu(S(h_{(2)})) \cdot \mu(h_{(3)}) \cdot b \cdot \mu(S(h_{(4)})) && (n = 4)\\
&= \mu(h_{(1)}) \cdot a \cdot \mu(S(h_{(2)})h_{(3)}) \cdot b \cdot \mu(S(h_{(4)})) && (n = 4)\\
&= \mu(h_{(1)}) \cdot a \cdot \epsilon(h_{(2)}) \cdot b \cdot \mu(S(h_{(3)})) && (n = 3)\\
&= \mu(h_{(1)}) \cdot a \cdot b \cdot \mu(S(h_{(2)})) && \text{Equation (1.1)}, (n = 2)\\
&= h \triangleleft (ab) = h \triangleleft m(a \otimes b),
\end{aligned}$$

$\square$

Let H be a Hopf algebra and $I \subseteq H$ a 2-sided ideal that is invariant under the adjoint action of H. Let $\mu : H \to A$ be a quantum moment map, and write $A \cdot \mu(I)$ or simply $A \cdot I$ for the left ideal of A generated by the image of I. The remainder of this section is devoted to an explanation of the following definition:

Definition 2.3. The **Hamiltonian reduction** of A by μ at I is defined as the algebra

$$A_I = (A/A \cdot \mu(I))^H.$$

We first argue that the H-action on A descends to $A/(A \cdot \mu(I))$, and hence it makes sense to consider the space of invariants. To see this, note that $\mu(I)$ is an H-submodule of A becasue I is H-invariant and μ is a map of H-modules. Next, Lemma 2.2 shows that the mulitplication map $m : A \otimes A \to A$ is H-linear, so the comppositio $A \otimes \mu(I) \to A \otimes A \to A$ is a map of H-modules. Therefore, the image of this composition is an H-submodule of A, and this image is precisely the ideal $A \cdot \mu(I)$. We conclude that the quotient $A/A \cdot \mu(I)$ carries an H-action.

Next, we argue that the algebra structure on A descends to A_I. We begin with a useful lemma:

Lemma 2.4. *Suppose $b \in A$ is such that $[b] \in (A/A \cdot \mu(I))^H$. Then, for any $x \in A \cdot \mu(I)$, the product xb lies in the ideal $A \cdot \mu(I)$.*

Proof. The condition $[b] \in (A/A \cdot \mu(I))^H$ implies that, for any $h \in H$, there is an $x \in A \cdot \mu(I)$ such that $h \triangleleft b = \epsilon(h)b + x$. To prove the lemma, it suffices to show that $\mu(h)b \in A \cdot \mu(I)$ for any $h \in I$. To this end, suppopse that $h \in I$. Then

$$\begin{aligned}
\mu(h)b &= \mu(h_{(1)}\epsilon(h_{(2)})) \cdot b = \mu(h_{(1)}) \cdot b \cdot \epsilon(h_{(2)})) && (n=2)\\
&= \mu(h_{(1)}) \cdot b \cdot \mu(S(h_{(2)})h_{(3)}) && (n=3)\\
&= \mu(h_{(1)}) \cdot b \cdot \mu(S(h_{(2)}) \cdot \mu(h_{(3)}) && (n=3)\\
&= (h_{(1)} \triangleleft b) \cdot \mu(h_{(2)}) && (n=2)\\
&= (\epsilon(h_{(1)}) \cdot b + x) \cdot \mu(h_{(2)}) && b \in (A/A \cdot \mu(I))^H, (n=2)\\
&= \epsilon(h_{(1)}) \cdot b \cdot \mu(h_{(2)}) + x' && (n=2)\\
&= b \cdot \mu(\epsilon(h_{(1)})h_{(2)}) + x' && (n=2)\\
&= b\mu(h) + x'
\end{aligned}$$

where $x' = x\mu(h_{(2)}) \in A$. In fact, $x' \in A \cdot \mu(I)$ because $x \in A \cdot \mu(I)$ and I is a 2-sided ideal of H. Now, $b\mu(h)$ is in $A \cdot \mu(I)$, and hence $\mu(h)b = b\mu(h) + x'$ belongs to $A \cdot \mu(I)$. $\square$

Proposition 2.5. *The multiplication on A descends to a well-defined associative algebra structure on $(A/A \cdot \mu(I))^H$.*

Proof. Abbreviate $A \cdot \mu(I)$ by $A \cdot I$. Lemma 2.4 implies that there is an induced map

$$m : (A/A \cdot I) \times (A/A \cdot I)^H \to A/A \cdot I$$

that makes the following diagram commute:

$$\begin{tikzcd}
A \otimes A \arrow[r] \arrow[d] & A \\
A \otimes (A/A \cdot I) \arrow[r] & A/A \cdot I \arrow[from=1-2, d] \\
A \otimes (A/A \cdot I)^H \arrow[r] \arrow[u, hook] \arrow[d] & A/A \cdot I \arrow[u, "="'] \\
(A/A \cdot I) \otimes (A/A \cdot I)^H \arrow[ur] &
\end{tikzcd},$$

where the top map is the multiplication on A, and the remaining maps are the obvious ones. It is enough to prove that the restriction of the map m to $(A/A{\cdot}I)^H \otimes (A/A{\cdot}I)^H$ lands in $(A/A{\cdot}I)^H$. To see this, suppose $a, b \in A$ are such that $[a], [b] \in (A/A \cdot I)^H$ and let $h \in H$. Then $h_{(1)} \triangleleft a = \epsilon(h_{(1)})a + x_a$ and $h_{(2)} \triangleleft b = \epsilon(h_{(2)})b + x_b$ for some x_a, x_b in $A \cdot I$, and

$$\begin{aligned}
h \triangleleft m(a,b) &= h \triangleleft [ab] = [h \triangleleft (ab)] = [(h_{(1)} \triangleleft a) \cdot (h_{(2)} \triangleleft b)] = [(\epsilon(h_{(1)})a + x_a)(\epsilon(h_{(2)})b + x_b)]\\
&= [\epsilon(h)ab + \epsilon(h_{(2)})x_a b + \epsilon(h_{(2)})a x_b + x_a x_b] = [ab],
\end{aligned}$$

where the last step uses Lemma 2.4. $\square$

Observe that there is an isomorphism $R : A/A \cdot I \xrightarrow{\sim} \mathrm{Hom}_A(A, A/A \cdot I)$ sending $[b] \in A/A \cdot I$ to the A-linear operator $R_{[b]} : [a] \mapsto [ab]$ of right multiplication by b.

Lemma 2.6. *There is an injective algebra homomorphism $(A/A\cdot I)^H \to \mathrm{End}_A(A/A\cdot I)^{op}$ making the following diagram commute:*

$$\begin{tikzcd} A/A\cdot I \arrow[r, "R"] & \mathrm{Hom}_A(A, A/A\cdot I) \\ (A/A\cdot I)^H \arrow[r] \arrow[u, hook] & \mathrm{End}_A(A/A\cdot I) \arrow[u, hook] \end{tikzcd}$$

Proof. First we show that if $[b] \in (A/A\cdot I)^H$, then the operator $R_{[b]} : A \to A/A\cdot I$ descends to $A/A\cdot I$. This follows from Lemma 2.4:

$$x \in A\cdot I \Rightarrow xb \in A\cdot I \Rightarrow R_{[b]}(x) = 0.$$

Thus we have an algebra map $(A/A\cdot I)^H \to \mathrm{End}_A(A/A\cdot I)^{\mathrm{op}}$. $\square$

Example: Characters of H. Let $\eta : H \to \mathbb{C}$ be an algebra homomorphism, i.e. a character of H. The kernel $I = \ker(\eta)$ is the 2-sided ideal generated by all elements of the form $h - \eta(h)$ for $h \in H$. In this case, we write A_η for the Hamiltonian reductin $A_{\ker(\eta)}$, and call it the Hamiltonian reduction of A by μ at the character η.

Lemma 2.7. *If $I = ker(\eta)$ is the kernel of a character η of H, then the map $(A/A\cdot I)^H \to \mathrm{End}_A(A/A\cdot I)^{op}$ is an isomorphism of algebras.*

Proof. Let $b \in A$ and suppose that $R_{[b]}$ descends to $A/A\cdot I$. We show that the image $[b]$ of b in $A/A\cdot I)$ lies in the space of invariants $(A/A\cdot I)^H$. By hypothesis, $xb \in A\cdot I$ for any $x \in A\cdot I$. For any $h \in H$, we have

$$\begin{aligned} [h \triangleleft b] &= [\mu(h_{(1)})b\mu(S(h_{(2)}))] \\ &= [(\mu(h_{(1)}) - \eta(h_{(1)}))b\mu(S(h_{(2)})) + \eta(h_{(1)})b\mu(S(h_{(2)})] \\ &= [\eta(h_{(1)})b\mu(S(h_{(2)})] \\ &= [\eta(h_{(1)})b(\mu(S(h_{(2)}) - \eta(S(h_{(2)}))) + \eta(h_{(1)})b\eta(S(h_{(2)}))] \\ &= [\eta(h_{(1)})b\eta(S(h_{(2)}))] \\ &= [\epsilon(h)b]. \end{aligned}$$

Therefore $[b] \in (A/A\cdot I)^H$. $\square$

Observe that there is a map $\mathrm{End}_A(A/A\cdot I) \to A/A\cdot I$ sending f to $f(1)$.

Question: Is the image of this map contained in $(A/A\cdot I)^H$?

If so, then there is an ismorphism $\mathrm{End}_A(A/A\cdot I)^{\mathrm{op}} \simeq (A/A\cdot I)^H$, as in [BFG, Section 3.4].

References

[BFG] R. Bezrukavnikov, M. Finkelberg, V. Ginzburg. Cherednik algebras and Hilbert schemes in characteristic p *Represent. Theory* 10 (2006), 254-298. With an appendix by Pavel Etingof.