Lecture notes: Tannaka duality for affine group schemes

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Abstract

These notes were written in preparation for the author's candidacy talk at the University of Texas at Austin. Tannaka duality for affine group schemes asserts that every rigid symmetric tensor category with a fiber functor is equivalent to the category of finite-dimensional representations of an affine group scheme. Moreover, one can recover the group through tensor automorphisms of the fiber functor. The talk will begin with the definition of an affine group scheme and representations thereof, discuss how extra structures on a coalgebra correspond to extra structures on its category of comodules, give outlines of the proofs of main statements of Tannaka duality, and provide examples. Time permitting, we mention recent results and current work in the area.

1 Introduction

We begin with a brief review of Pontryagin duality. Let G be a locally compact abelian group and define the dual group $\hat{G}$ of G as the group of unitary characters of G. In other words, $\hat{G}$ consists of the continuous homomorphisms from G to the unit circle S^1. The main statement of Pontryagin duality is that the homomorphism $G \to \hat{\hat{G}}$ given by evaluation is an isomorphism [7]. In this way, one 'recovers' the group G from its dual $\hat{G}$ by taking the double dual. Moreover, properties of G are reflected in properties of $\hat{G}$ and vice versa. For example, G is compact if and only if $\hat{G}$ is discrete.

As we will see, the statements of Pontryagin duality bear some similarity with results concerning affine group schemes. Specifically, instead of the dual group $\hat{G}$, we consider the category of $\mathsf{Rep}(G)$ of finite dimensional representations of an affine group scheme G over a field k and show that one can recover G from $\mathsf{Rep}(G)$ and the forgetful functor to vector spaces over k. We characterize the categories that appear as representations of affine groups, and see how properties of G are reflected in properties of the category $\mathsf{Rep}(G)$.

Notation: We fix a commutative ring k; eventually we require k to be a field. Let CAlg_k be the category of commutative algebras over k, and R will always denote an object of CAlg_k. Let Set and Grp denote the categories of sets and groups, respectively. For a field k, let $\underline{\mathsf{Vec}}_k$ (resp. Vec_k) denote the category of vector spaces (resp. finite dimensional vector spaces) over k. For a k-coalgebra A, let $\underline{\mathsf{Com}}(A)$ (resp. $\mathsf{Com}(A)$) denote the category of right A-comodules (resp. finite dimensional right A-comodules). We write h^X for the representable functor $\mathrm{Hom}(X, -)$ defined by an object X of a category.

2 Affine group schemes

Definition: An **affine group scheme** is a representable functor $G : \mathsf{CAlg}_k \to \mathsf{Set}$ that factors through Grp. Equivalently, an affine group scheme is a group object in the category $\mathsf{Func}(\mathsf{CAlg}_k, \mathsf{Set})$

that is representable. The **coordinate ring** $\mathcal{O}(G)$ of an affine group G is the commutative k-algebra that represents G.

There is a Hopf algebra structure on $\mathcal{O}(G)$, obtained by applying the Yoneda lemma to the multiplication, unit, and inversion maps on G. Conversely, if $A \in \mathsf{CAlg}_k$ is a Hopf algebra, then $h^A = \mathrm{Hom}(A, -)$ is an affine group. An altenative view of affine group schemes is as the group objects in the category of affine schemes over k, which are precisely the spectra of Hopf algebras. An advantage to the approach chosen here is that affine group schemes can be compared to any functor $\mathsf{CAlg}_k \to \mathsf{Set}$, not just ones that define a scheme.

Now we describe the notion of representation of an affine group scheme on a module for the base ring. Let V be a k-module and define a functor $\mathrm{Aut}_V : \mathsf{CAlg}_k \to \mathsf{Grp}$ by $\mathrm{Aut}_V(R) = \mathrm{Aut}_{R-\mathrm{lin}}(V \otimes R)$. If V is free of rank n, then $\mathrm{Aut}_V \simeq \mathrm{GL}_n$. A **representation** of an affine group scheme G on V as a natural transformation $r : G \to \mathrm{Aut}_V$ of group-valued functors.

Theorem 1. *There is a canonical equivalence between the category of representations of G and the category of $\mathcal{O}(G)$-comodules.*

This theorem is not surprising since a contravariant functor takes $\mathcal{O}(G)$ to G, and it takes a bit of caution to make this intuition precise. From this result, we see that in order to understand representations of affine group schemes, it is helpful to think about categories of comodules, which is what we do in the next section.

3 Categories of comodules

Let A be a k-coalgebra, and let $\omega : \mathsf{Com}(A) \to \mathsf{Vec}_k$ be the forgetful functor from the category of finite dimensional A-comodules. The goal of this section is to relay two messages. First, we can recover A as a coalgebra from $\mathsf{Com}(A)$ and ω. The key step is:

Proposition 2. *There are isomrphisms*

$$\mathrm{Hom}_{k-\mathrm{lin}}(A, V) \simeq \mathrm{Nat}(\omega, \omega \otimes V)$$

functorial in $V \in \underline{\mathsf{Vec}}_k$, where $\omega \otimes V$ denotes the composition of functors $(- \otimes V) \circ \omega$.

The second message is that there is an exact correspondence between certain extra structures on the coalgebra A and certain extra structures on the abelian category $\mathsf{Com}(A)$, given in the following chart:

Structure on A	Structure on $\mathsf{Com}(A)$
bialgebra	tensor
commutative bialgebra	symmetric tensor
Hopf algebra	rigid tensor
commutative Hopf algebra	rigid symmetric tensor

Caveats: The tensor structure on $\mathsf{Com}(A)$ must be such that ω is a tensor functor, and if the tensor structure is symmetric, then it must descend to the usual symmetric structure on Vec_k via ω. In justifying the chart, it is usually easier to prove that a structure on A is reflected on the category of comodules; the other direction is slightly more subtle.

4 Recovering G from $\mathsf{Rep}(G)$.

Let $\mathsf{Rep}(G)$ denote the category of finite dimensional representations of an affine group G. We have observed that $\mathsf{Rep}(G)$ is equivalent to the category of finite dimensional comodules for the commutative Hopf algebra $\mathcal{O}(G)$. Therefore, $\mathsf{Rep}(G)$ is rigid symmetric tensor category. Let $\omega : \mathsf{Rep}(G) \to \mathsf{Vec}_k$ be the forgetful functor. In this section, we explain one part of Tannaka duality for affine group schemes: one can recover an affine group scheme from its category of finite dimensional representations and the forgetful functor ω.

Define the group-valued functor $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on CAlg_k as follows. Let R a k-algebra and consider the composition:

$$\omega \otimes R : \mathsf{Rep}(G) \xrightarrow{\omega} \mathsf{Vec}_k \xrightarrow{-\otimes R} \mathsf{Mod}(R).$$

It is clear that $\omega \otimes R$ is a strict tensor functor. (Recall the $\mathsf{Mod}(R)$ has a tensor structure since R is commutative.) Define $\underline{\mathrm{Aut}}^{\otimes}(\omega)(R)$ as the group of automorphisms of $\omega \otimes R$ as a tensor functor.

Theorem 3. *There is a canonical isomorphism* $G \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega)$.

The steps of the proof are as follows. First, $\underline{\mathrm{Aut}}^{\otimes}(\omega) = \underline{\mathrm{End}}^{\otimes}(\omega)$ since $\mathsf{Rep}(G)$ is rigid. Second,

$$\underline{\mathrm{End}}(\omega)(R) \simeq \mathrm{Nat}_{R-\mathrm{lin}}(\omega \otimes R, \omega \otimes R) \simeq \mathrm{Nat}_{k-\mathrm{lin}}(\omega, \omega \otimes R) \simeq \mathrm{Hom}_{k-\mathrm{lin}}(A, R)$$

where the last isomorphism comes from the propostion in the previous section. Finally, one must argue that tensor endomorphisms correspond to algebra homomorphisms, i.e. $\mathrm{End}^{\otimes}(\omega \otimes R) \simeq \mathrm{Hom}_{k-\mathrm{alg}}(A, R)$, from which it is immediate that $\underline{\mathrm{End}}^{\otimes}(\omega)(R) = G(R)$.

Corollary 4. *Let G and H be affine group schemes over k. There is a bijection between morphisms of k-schemes $G \to H$ and symmetric tensor functors $\mathsf{Rep}(H) \to \mathsf{Rep}(G)$ that commute with the forgetful functors.*

5 Neutral Tannakian categories

In this section, C denotes an essentially small abelian k-linear category. A **fiber functor** on C is defined as a faithful exact k-linear functor $\mathsf{C} \to \mathsf{Vec}_k$. For example, if A is a k-coalgebra, then the forgetful functor $\mathsf{Com}(A) \to \mathsf{Vec}_k$ is a fiber functor. The first goal of this section is to prove that every fiber functor appears in this way.

Theorem 5. *The category C is equivalent to $\mathsf{Com}(A)$ for a k-coalgebra A if and only if C admits a fiber functor ω.*

We give an outline of a proof due to Serre [8].

- For an object X of C, let $\langle X \rangle$ denote the full subcategory of C whose objects are the quotients of subobjects of direct sums of copies of X.
- For a fixed (but arbitrary) object X, find a projective generator P_X of $\langle X \rangle$ such that $\omega|\langle X \rangle \simeq h^{P_X}$.

- By standard arguments (e.g. Barr-Beck), we have that

$$\langle X \rangle \simeq \mathsf{Mod}(\mathrm{End}(P_X)) \simeq \mathsf{Com}(A_X)$$

  where A_X is the finite dimensional k-coalgebra $\mathrm{End}(P)^\vee$.

- Note $\mathrm{End}(P)$ is finite dimensional since ω is faithful. This fact is crucial here since the dual of an algebra is not naturaly a coalgebra in infinite dimensions. Also, using the Yoneda lemma, $A_X = \mathrm{End}(P)^\vee \simeq (\mathrm{End}(\omega|\langle X \rangle)^{\mathrm{op}})^\vee$.

- If $\langle X \rangle \subset \langle Y \rangle$, then restriction gives a k-algebra homomorphism

$$\mathrm{End}(\omega|\langle Y \rangle) \to \mathrm{End}(\omega|\langle X \rangle).$$

  Dualizing gives a k-coalgebra homomorphism $A_X \to A_Y$.

- Let $A = \varinjlim A_X$ and argue that $\mathsf{C} \simeq \mathsf{Com}(A)$

The theorem and results from previous section immediately imply the following corollary, which characterizes categories that appear as representations of affine group schemes and forms another part of Tannaka duality for affine group schemes.

Corollary 6. *Suppose that* C *is a symmetric rigid tensor category. Then* C *is equivlent to the category of representations of an affine group* G *if and only if* C *admits a symmetric tensor fiber functor* ω*.*

Definition: **neutral Tannakian category** is a symmetric rigid tensor category C equipped with a symmetric tensor fiber functor $\omega : \mathsf{C} \to \mathsf{Vec}_k$. The corollary implies that every neutral Tannakian category C is equivalent to $\mathsf{Rep}(G)$ for some affine group G, called the **Tannakian fundamental group** of C.

6 Examples

1. Let Γ be any group. The category $\mathsf{Rep}(\Gamma)$ of finite dimensional representations of Γ is a neutral Tannakian category. Its Tannakian fundamental group is called the **algebraic hull** of Γ, denoted Γ^{alg}.

2. If Γ is finite in the previous example, then the algebraic hull of Γ is the affine group scheme corresponding to the Hopf algebra $\mathrm{Fun}(\Gamma)$ of k-valued functions on Γ, that is, $\Gamma^{\mathrm{alg}} = h^{\mathrm{Fun}(\Gamma)}$. For any commutative k-algebra R with no idempotents other than 0 and 1, $\Gamma^{\mathrm{alg}}(R) = \Gamma$. To see this, one exploits that the fact that the delta functions δ_γ in $\mathrm{Fun}(\Gamma)$ are idempotents whose sum is the unit element of $\mathrm{Fun}(\Gamma)$.

3. Let Γ be a finite group. Let Vec_Γ denote the category of Γ-graded vector spaces with tensor product given as follows: for objects V and W of Vec_Γ, define the g-homogeneous component of $V \otimes W$ as $(V \otimes W)_g = \sum_{xy=g} V_x \otimes_k W_y$. It is easy to see that the forgetful functor ω on Vec_Γ is a tensor fiber functor. However, it is not symmetric unless Γ is commutative. Define an object $X = \bigoplus_{g \in \Gamma} X_g$ of Vec_Γ by setting $X_g = k$ for all $g \in G$. One can verify that $\mathsf{Vec}_\Gamma = \langle X \rangle$

and that $\mathrm{End}(\omega)$ is isomorphic to the commutative Hopf algebra $\mathrm{Fun}(\Gamma)$ of functions on Γ. By the proof of the theorem in the previous section, we can identify

$$\mathsf{Vec}_\Gamma \simeq \mathsf{Mod}(\mathrm{Fun}(\Gamma)) \simeq \mathsf{Com}(k[\Gamma]),$$

where $k[\Gamma]$ is the group algebra of Γ, which is dual to $\mathrm{Fun}(\Gamma)$.

4. Let X be a connected semi-locally simply connected topological space and LS_X the category of local systems on X, that is, locally constant sheaves of finite-dimensional complex vector spaces. The standard tensor product of sheaves gives LS_X the structure of a rigid symmetric tensor category. Fix a point $x \in X$ and consider the functor $\omega : \mathrm{LS}_X \to \mathsf{Vec}_k$ that sends the local system $\mathcal{F}$ to the stalk $\mathcal{F}_x$ at x. This is a fiber functor, and makes LS_X a neutral Tannakian category. The Tannakian fundamental group is the algebraic hull of $\pi_1(X, x)$.

7 Dictionary between properties of G and $\mathsf{Rep}(G)$.

In the introduction, we mentioned that properties of a locally compact abelian group G are reflected in properties of its dual $\hat{G}$. There is a similar sort of 'dictionary' between properties of an affine group scheme G and the category $\mathsf{Rep}(G)$. We give some instances of this dictionary; this involves introducing some terminology.

An affine group scheme G is **finite** if $\mathcal{O}(G)$ is a finite dimensional k-vector space. An affine group scheme G is **algebraic** if $\mathcal{O}(G)$ is a finitely generated k-algebra. An affine group scheme G is **proreductive** if it a projective limit of reductive groups. For an object X of a tensor category $(\mathsf{C}, \otimes)$, let $\langle X \rangle_\otimes$ denote the full subcategory of quotients of subobjects of direct sums of $X^{\otimes n}$ for varying n. An abelian category is **semisimple** if every object decomposes as a direct sum of simples.

Proposition 7. *Let G be an affine group scheme.*

1. *G is finite if and only if $\mathsf{Rep}(G) = \langle V \rangle$ for some object V.*
2. *G is algebraic if and only if $\mathsf{Rep}(G) = \langle V \oplus V^\vee \rangle_\otimes$ for some object V.*
3. *G is proreductive if and only if $\mathsf{Rep}(G)$ is a semisimple category.*

8 Further directions and applications

Recall that a fiber functor is valued in the category of vector spaces over k, equivalently, the category of quasi-coherent sheaves on the affine scheme $\mathrm{Spec}(k)$. Replacing $\mathrm{Spec}(k)$ with an arbitrary k-scheme, we obtain the following generalization of a neutral Tannakian category: A **(general) Tannakian category** is a symmetric rigid tensor category C with $\mathrm{End}_{\mathsf{C}}(1) \simeq k$ equipped with a faithful exact symmetric tensor functor $\omega : \mathsf{C} \to \mathsf{QC}(S)$ from C to the category of quasicoherent sheaves on a k-scheme S. The hypotheses imply that ω is valued in locally free sheaves of finite rank. In [3], Deligne demonstrates that every Tannakian category is equivalent to the category of representations of an affine groupoid scheme acting on S

We describe recent work by Brandenburg and Chirvasitu [2]. Let X be a quasi-separated scheme and Y an arbitrary scheme. A morphism $Y \to X$ induces a pullback functor $f^* : \mathsf{QC}(X) \to \mathsf{QC}(Y)$ on the categories of quasi-coherent sheaves. This functor is tensor and cocontinuous, i.e. it preserves direct sums, cokernels, and tensor products. Brandenburg and Chirvasitu [2] have shown that the construction $f \mapsto f^*$ defines an equivlance between the category of morphisms of schemes $Y \to X$ and the category of cocontinuous tensor functors $\mathsf{QC}(X) \to \mathsf{QC}(Y)$; in symbols:

$$\mathrm{Hom}_{\mathsf{Sch}}(Y, X) \simeq \mathsf{Func}_{c\otimes}(\mathsf{QC}(X), \mathsf{QC}(Y)).$$

This is analogous to a result given in this talk, namely that there is a bijection between homomorphisms between affine group schemes and tensor functors between their categories of representations. It is not known if the result holds for quasi-compact, quasi-separated algebraic stacks.

Finally, we note an appearance of Tannakian categories in the geometric Langlands program [5]. Let G be a semisimple group and G^L its Langlands dual. A well-known result is that the Hecke algebra $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$ is isomorphic to the complexified Grothendieck ring $\mathbb{C} \otimes_{\mathbb{Z}} K(\mathsf{Rep}(G^L)$. However, passing to the Grothendieck ring looses much information about G^L. Instead, consider the category $P(\mathrm{Gr})$ of of semisimple $G(\bar{\mathbb{F}}_p[[z]])$-equivariant perverse sheaves on $\mathrm{Gr} = G(\bar{\mathbb{F}}_p((z)))/G(\bar{\mathbb{F}}_p[[z]])$.

Theorem 8 (Ginzburg)**.** *The category $P(\mathrm{Gr})$ is a semismiple neutral Tannakian category with fiber functor given by taking hyper-cohomology. Moreover, the Tannakian fundamental gorup of $P(\mathrm{Gr})$ is G^L.*

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