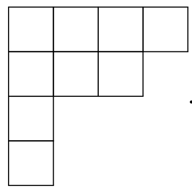
Notes for 'Young Tableaux and Symmetric Polynomials'[1]

Iordan Ganev

1. Young tableaux

A **partition** of a nonnegative integer n is a weakly decreasing sequence of positive integers whose sum is n. A partition is usually denoted $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. For example, (4,3,1,1) is a partition of 9. The only partition of 0 is the empty partition $\emptyset$. A **Young diagram** is a collection of boxes arranged in left-justified rows such that the number of boxes in each row decreases weakly from top to bottom. An example is



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The **size** of a Young diagram is the total number of boxes. Observe that partitions of n are in one-to-one correspondence with Young diagrams of size n. If λ is a partition, then the corresponding Young diagram is said to have **shape** λ. A **Young tableau** is a Young diagram that is filled by positive integers according to two rules: (1) the entries in each row are weakly increasing and (2) the entries in each column are strictly increasing. An example is

1	1	3	4
2	4	4	
4			
6			

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2. Bumping and products

The **bumping** algorithm takes a tableau T and a positive integer x and produces a new tableau, which we denote $(T \leftarrow x)$. The idea is to give a systematic way of adding to T a new box with entry x. I will illustrate this through an example during the talk, but here is the full algorithim written out. If x is greater than or equal to all the entries in the first row of T, then add a box labelled by x at the end of the first row. Otherwise, find the leftmost box of the first row that has entry strictly greater than x. Relabel this box with x and remove ('bump') the old entry, call it y. Now repeat the process with y and the second row. Keep going until the bumped entry can be placed in a new box at the end of a row, possibly at the very bottom of the diagram.

Let T and U be two tableaux. Let $x_1, x_2, \ldots, x_k$ denote the entries of U, listed row by row from bottom to top, and left to right in each row. Define the **product** $T \cdot U$ of T and U to be the new tableau obtained by successive bumping:

$$T \cdot U = (\ldots(((T \leftarrow x_1) \leftarrow x_2) \leftarrow x_3) \cdots \leftarrow x_k).$$

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Since each step adds one new box, the size of the product of T and U is the sum of the sizes of T and U. Here are some important properties, whose proofs are exercises:

- Let $\emptyset$ denote the empty tableau. For any tableau T, we have $T \cdot \emptyset = \emptyset \cdot T = T$, so $\emptyset$ is the **identity** of this multiplication.
- The multiplication is **associative**: $T \cdot (U \cdot V) = (T \cdot U) \cdot V$ for any tableaux T, U, V.
- The multiplication is not commutative: in general $T \cdot U \neq U \cdot T$.

In fancy language, this shows that the set of tableaux, together with this product, forms a **monoid**. (A monoid M is a set with an associative multiplication $\cdot : M \times M \to M$ and a unit. So a monoid is like a group, but without the requirement of inverses.)

3. Schur polynomials

I will assume familiarity with rings. Arguably the most famous ring is the integers $\mathbb{Z}$. Another important ring is $\mathbb{Z}[x]$, the collection of polynomials in the variable x with integer coefficients. For the remainder of this section, we fix a positive integer n and consider the ring $\mathbb{Z}[x_1, \ldots, x_n]$ of polynomials in n variables with integer coefficients.

Let T be a tableau whose entries are in the set $\{1, \ldots, n\}$. Let m_i be the number of boxes of T that are labelled with the integer i, where $1 \leq i \leq n$. Associate to T the monomial

$$x^T = x_1^{m_1} \cdots x_n^{m_n}.$$

I will do an example during the talk. Now consider a partition λ of k. To λ we associate a polynomial s_λ in $\mathbb{Z}[x_1, \ldots, x_n]$, called the **Schur polynomial** of λ, and defined as

$$s_\lambda(x_1, \ldots, x_n) = \sum_{T \text{ has shape } \lambda} x^T,$$

where the sum runs only over tableau with entries in $\{1, \ldots, n\}$.

During the talk, we will see that the Schur polynomial of the partition $\lambda = (k)$ is the k-th **complete symmetric polynomial** in n variables, defined as

$$h_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

Some examples are $h_1(x_1, x_2) = x_1 + x_2$ and $h_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$. As an exercise, write out h_1, h_2, and h_3 for $n = 3, 4$.

The Schur polynomial of the partition $\lambda = (1^k) = (1, \ldots, 1)$ of k is the k-th **elementary symmetric polynomial** in n variables, defined as

$$e_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

for $k \geq 1$ and $e_0 = 1$ for $k = 0$. Observe that the difference between the definitions of h_k and e_k is that weak inequalities are now replaced by strict inequalities. This difference reflects the rules for filling in the rows and columns of a Young diagram in order to obtain a Young tableau. The complete symmetric polynomials are 'bigger' than the elementary symmetric polynomials in the sense that any monomial that appears in e_k will also appear in h_k. Some examples are $e_1(x_1, x_2) = x_1 + x_2$, $e_2(x_1, x_2) = x_1 x_2$, and $e_k(x_1, x_2) = 0$ if $k > 2$. As an exercise, write out e_k for $n = 3, 4$. Note that $e_k = 0$ if $k > n$.

4. Symmetric polynomials

A polynomial f in $\mathbb{Z}[x_1, \ldots, x_n]$ is said to be **symmetric** if it is fixed by any permutation of the variables. In symbols, this says that

$$f(x_1, x_2, \ldots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$

for any permuation σ. (A permuation is just a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.) To see if you understand this defintion, convince yourself that each e_k and h_k is a symmetric polynomial. In fact, the Schur polynomial s_λ is always symmetric, but I will not prove this.

The collection $\Lambda^{(n)}$ of symmetric polynomials in n variables forms a subring of $\mathbb{Z}[x_1, \ldots, x_n]$. It is a theorem that the elementary symmetric polynomials $\{e_k\}_{k=1}^n$ generate $\Lambda^{(n)}$ as a $\mathbb{Z}$-algebra. Moreover, these e_k are algebraically independent, so we can write $\Lambda^{(n)} \simeq \mathbb{Z}[e_1, \ldots, e_n]$. The same result is true for $\{h_k\}_{k=1}^n$. See [4, pp. 20-22] for more details.

Define a ring homomorphism $\rho_n : \Lambda^{(n+1)} \to \Lambda^{(n)}$ by evaluating at $x_{n+1} = 0$ (that is, remove all monomials in which x_{n+1} appears). So we have an 'inverse system':

$$\Lambda^{(0)} \xleftarrow{\rho_0} \Lambda^{(1)} \xleftarrow{\rho_1} \Lambda^{(2)} \xleftarrow{\rho_2} \Lambda^{(3)} \xleftarrow{\rho_3} \cdots.$$

The inverse limit Λ of this system is called the **ring of symmetric functions**. In symbols,

$$\Lambda = \varprojlim \Lambda^{(n)} = \{(f_n)_{n \in \mathbb{N}_0} \in \prod_{n \in \mathbb{N}_0} \Lambda^{(n)} : \rho_n(f_{n+1}) = f_n\}.$$

This construction may seem very bizarre if you haven't seen inverse limits before. The advantage of the ring Λ is that it allows us to discuss symmetric polynomials in arbitrarily many variables. In some cases, it serves as a 'bookkeeping' device that simplifies more complicated structures.

5. Application: the representation theory of S_n

Most likely, I will not have time to say much about these last two sections. Besides, the material discussed below is mostly beyond the undergraduate level. Still, I would like students to be aware some of the applications of Young tableaux and symmetric polynomials.

A **representation** of a finite group G is a vector space V, together with a homomorphism $\rho : G \to \mathrm{GL}(V)$. In other words, a representation of G a linear action of G on V. A subspace W of a representation V is a **subrepresentation** if W is invariant under the action of G; in symbols, if $\rho(g)(w) \in W$ for any $g \in G$ and $w \in W$. A representation V of G is **irreducible** if it has no nontrvial subrepresentations. One should think of irreducible representations as the building blocks for all other representations. It is a theorem from basic representation theory that the number of irreducible representations of a finite group G is equal to the number of conjugacy classes of G. In general there is no 'natural' way to associate to each conjugacy class an irreducible representation of G.

In the case of the symmetric group S_n, however, something special happens. Recall that a conjugacy class of S_n consists of all the elements with a certain cycle type decomposition. In turn, it is not difficult to see that cycle type decompositions are in bijection with partitions of n. Thus, to every conjugacy class in S_n we can use the arguments above to obtain a Schur polynomial s_λ. On the other hand, let $\mathcal{R}_n$ be the free $\mathbb{Z}$-module generated by the irreducible representations of S_n. Set $\mathcal{R} = \bigoplus_n \mathcal{R}_n$. There is a meaningful way to define (1) a ring structure on $\mathcal{R}$ and (2) and isomorphism $\mathcal{R} \xrightarrow{\sim} \Lambda$ (See [1, p.294]). Moreover, under this isomorphism, the irreducible representations are mapped to the Schur polynomials. In particular, the alternating representation of S_k corresponds to e_k, while the trivial representation of S_k corresponds to h_k. Thus, consideration of symmetric polynomials gives a natural bijection between conjugacy classes in S_n and irreducible representations of S_n.

6. Application: the cohomology of Grassmannians

Perhaps you are aware that cohomology is a machine that assigns to each topological space X a (graded) ring $H^*(X)$. An important space in geometry is the **Grassmannian** of k-dimensional subspaces in $\mathbb{C}^n$, defined as

$$\mathrm{Gr}(k,n) = \{W : \; W \text{ is a subspace of } \mathbb{C}^n \text{ and } \dim(W) = k\}.$$

One description of the cohomology ring $H^*(\mathrm{Gr}(k,n))$ of the Grassmannian involves Young tableaux. First, for $0 \leq i \leq n$, fix a subspace F_i of $\mathbb{C}^n$ such that F_i has dimension i and $F_i \subset F_{i+1}$. We obtain what is known as a **complete flag**

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathbb{C}^n.$$

Let λ be a partition of n whose Young diagram has at most k rows and at most $n-k$ columns. Define the **Schubert variety** Ω_λ corresponding to λ as

$$\Omega_\lambda = \{W \in \mathrm{Gr}(k,n) : \dim(W \cap F_{n+i-\lambda_i}) \geq i, \; \text{for } 1 \leq i \leq k\}.$$

It turns out that each Schubert variety defines an element σ_λ of the cohomology ring $H^*(\mathrm{Gr}(k,n))$. In fact, one can show that the σ_λ form a $\mathbb{Z}$-basis for $H^*(\mathrm{Gr}(k,n))$ and their multiplication satisfies the same formulas that the Schur polynomials s_λ satisfy (see [2, p. 146] for a more complete discussion).

References

[1] D. Bump. *Lie Groups.* Springer, 2004.

[2] W. Fulton. *Young Tableaux.* London Mathematical Society Student Texts, 1997.

[3] W. Fulton and J. Harris. *Representation Theory.* Springer, 2004.

[4] I.G. MacDonald. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 1999.