Notes for 'An overview of Springer Theory: Part II'

Iordan Ganev

18 October 2012

1 Sheaf-theoretic set-up

Throughout, $\mathsf{Shv}(X)$ denotes the category of sheaves of complex vector spaces on a space X. A **local system** on X is a locally constant sheaf of finite-dimensional complex vector spaces. The category of local systems on X is denoted $\mathsf{LS}(X)$.

1.1 The dualizing complex

In the last talk, we considered the Borel-Moore homology of a complex variety X. Our first goal in this talk is to describe Borel-Moore homology as sheaf cohomology with coefficients in a specific sheaf, called the dualizing sheaf.

Recall that all spaces X we are interested in admit a closed embedding into a smooth variety (or manifold, depending on the context) M:

$$i : X \hookrightarrow M.$$

For a sheaf $\mathcal{F}$ on M and on open subset U of M, denote the sections over U supported on X as

$$\Gamma_{[X]}(U, \mathcal{F}) = \{f \in \Gamma(U, \mathcal{F}) \mid \operatorname{supp}(f) \in X\}.$$

The functor $\Gamma_{[X]}$ of taking global sections supported on X is left exact. Its right derived functors at the constant sheaf compute the cohomology of M relative to the complement of X, and this is precisely how we thought of the Borel-Moore homology of X:

$$R^k\Gamma_{[X]}(\mathbb{C}_M) = H^k(M, M \setminus X) = H_{m-k}(X).$$

Here $m = \dim_{\mathbb{R}}(M)$ and $\mathbb{C}_M$ denotes the constant sheaf on M, i.e. the pullback of $\mathbb{C}$ under the map from M to a point.

Define $i^! : \mathsf{Shv}(M) \to \mathsf{Shv}(X)$ by

$$\Gamma(V, i^!\mathcal{F}) = \lim_{U} \Gamma_{[X]}(U, \mathcal{F})$$

where V is on open subset of X and the limit is over open subsets U of M that contain V, directed by inclusion. The functor $i^!$ is left exact and we have

$$H^p(X, Ri^!(\mathbb{C}_M[m])) = H_{-p}(X).$$

In words, the derived functor $Ri^!$ of $i^!$ computes Borel-Moore homology after shifting the constant sheaf $\mathbb{C}_M$ in the derived category by the real dimension of M. We see that the sheaf $Ri^!(\mathbb{C}_M[m]))$ of X plays a special role, and therefore we give it a special name:

Definition. The **dualizing complex** on X is defined as $\mathbb{D}_X = Ri^!(\mathbb{C}_M[m]))$.

To be clear, the dualizing complex is an object of the bounded derived category $D^b(\mathsf{Shv}(X))$ of sheaves on X, and it has the special property that $H^p(X, \mathbb{D}_X) = H_{-p}(X)$.

1.2 Constructible complexes

Let X be a complex algebraic variety. A sheaf $\mathcal{F}$ on X is called **constructible** if there exists a finite algebraic stratification $X = \coprod X_\alpha$ of X such that for each α, the stratum X_α is a locally closed smooth algebraic subvariety of X and the restriction of $\mathcal{F}$ to X_α is a local system. A possibly helpful diagram to consider is the following:

$$\begin{tikzcd} \mathsf{Shv}(X) \arrow[r, "\text{res}"] & \mathsf{Shv}(X_\alpha) \\ \{\text{constructible sheaves}\} \arrow[u, hook] \arrow[r, dashed] & \mathsf{LS}(X_\alpha) \arrow[u, hook] \end{tikzcd}$$

Let $D^b(\mathsf{Shv}(X))$ denote the bounded derived category of sheaves on X. An object A of $D^b(\mathsf{Shv}(X))$ is said to be a **constructible complex** if all the cohomology sheaves $\mathcal{H}^i(A)$ are constructible. Let $D^b(X)$ be the full subcategory of $D^b(\mathsf{Shv}(X))$ consisting of constructible complexes. (Warning: $D^b(X)$ is not the derived category of constructible sheaves.) The prototypical example of a constructible complex is the de Rham complex on X. The actual sheaves in the complex are 'big', while the cohomology is locally constant on strata so is in some sense more manageable.

1.3 Verdier duality

We return to the case of closed embedding $i : X \hookrightarrow M$. The functor $Ri^!$ takes $D^b(M)$ to $D^b(X)$. In particular, the dualizing complex $\mathbb{D}_X$ is constructible. The Verdier duality functor $D^b(X) \to D^b(X)$ is defined as

$$A \mapsto A^\vee = \mathcal{H}om(A, \mathbb{D}_X),$$

where $\mathcal{H}om$ denotes the internal hom in $D^b(X)$. Here we use the property that the internal hom in the entire derived category $D^b(\mathsf{Shv}(X))$ of two constructible complexes is again constructible.

Some basic properties of the Verdier duality functor are that $(\mathbb{C}_X)^\vee = \mathbb{D}_X$ and $(\mathcal{F}^\vee)^\vee = \mathcal{F}$.

For an algebraic map $f : X \to Y$, let f_* and f^* denote the derived functors of sheaf-theoretic direct and inverse image, respectively. These take constructible complexes to constructible complexes and form an adjoint pair (f^*, f_*). There is another pair of adjoint functors $(f_!, f^!)$ defined using Verdier duality:

$$f_!A = (f_*(A^\vee))^\vee \qquad\qquad f^!B = (f^*(B^\vee))^\vee$$

for an object A in $D^b(X)$ and an object B in $D^b(Y)$. These functors enjoy the following properties

- $H^\bullet(Y, f_*A) = H^\bullet(X, A)$
- $f^*(\mathbb{C}_Y) = \mathbb{C}_X$

- $f^!(\mathbb{D}_Y) = \mathbb{D}_X$
- If f is proper, then $f_! = f_*$.
- If f is smooth with smooth fibers of real dimension d, then $f^! = f^*[d]$.
- For a closed embedding $i : X \hookrightarrow M$, the previous definition of $i^!$ coincides with the new one.
- Base change: For a Cartesian square

$$\begin{tikzcd} X \times_W Y \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & W \end{tikzcd} .$$

  we have $g^! f_* = \tilde{f}_* \tilde{g}^!$

2 The convolution algebra and the Ext algebra

Let M be a smooth complex variety with $m = \dim_{\mathbb{R}}(M)$ and $\mu : M \to N$ a proper map to a (not necessarily smooth) variety N. Let $Z = M \times_N M$ be the fiber product, and recall from last time that the Borel-Moore homology $H_*(Z)$ has a convolution algebra structure, and $H_m(Z)$ is a subalgebra. We abbreviate $H_m(Z)$ by $H(Z)$. The example to keep in mind is the Springer resolution $\mu : \tilde{N} \to N$, where $Z = \tilde{N} \times_N \tilde{N}$ is the Stienberg variety.

If A and B are complexes in $D^b(X)$, their Ext groups are defined as shifted homs:

$$\mathrm{Ext}^k_{D^b(X)}(A, B) = \mathrm{Hom}_{D^b(X)}(A, B[k]).$$

We can also express the Ext groups in terms of cohomology with coefficients in the internal hom sheaf:

$$\mathrm{Ext}^k_{D^b(X)}(A, B) = H^k(X, \mathcal{H}om(A, B)).$$

Set $\mathcal{L} = \mu_*(\mathbb{C}_M[m]) \in D^b(N)$. The main result of this section is an isomorphism between the convolution algebra $H_*(Z)$ and the Ext algebra of $\mathcal{L}$:

Proposition 1. *There is a (not necessarily grading-preserving) isomorphism of algebras*

$$H_*(Z) \simeq \mathrm{Ext}^*_{D^b(N)}(\mathcal{L}, \mathcal{L}).$$

Moreover, $H(Z) \simeq \mathrm{End}_{D^b(N)}(\mathcal{L})$.

Note: In my notes on representations of finite groups for the Eugene workshop, we demonstrate that a certain convolution algebra of functions on a product $X \times X$ of a finite set with itself is isomorphic to a matrix algebra. The present result is a generalization (in the appropriate sense) of the result about finite sets.

Proof. (sketch) Consider the Cartesian square

$$\begin{tikzcd}
Z = M \times_N M \arrow[r, "p_2"] \arrow[d, "p_1"'] & M \arrow[d, "\mu"'] \\
M \arrow[r, "\mu"] & N
\end{tikzcd} .$$

In this proof, we ignore gradings as well as shifts in the derived category. We have

$$\begin{aligned}
\mathrm{Ext}^*_{D^b(N)}(\mathcal{L}, \mathcal{L}) &= \mathrm{Ext}^*_{D^b(N)}(\mu_*(\mathbb{C}_M), \mu_*(\mathbb{C}_M)) \\
&= \mathrm{Ext}^*_{D^b(N)}(\mu_!(\mathbb{C}_M), \mu_*(\mathbb{C}_M)) \quad \text{since } \mu \text{ is proper} \\
&= \mathrm{Ext}^*_{D^b(M)}(\mathbb{C}_M, \mu^! \mu_*(\mathbb{C}_M)) \quad \text{by adjunction} \\
&= H^*(M, \mathcal{H}om(\mathbb{C}_M, \mu^! \mu_*(\mathbb{C}_M)) \\
&= H^*(M, \mu^! \mu_*(\mathbb{C}_M)) \quad \text{since } \mathcal{H}om \text{ out of the constant sheaf does nothing} \\
&= H^*(M, (p_1)_*(p_2)^!(\mathbb{C}_M)) \quad \text{by base change} \\
&= H^*(Z, (p_2)^!(\mathbb{C}_M)) \quad \text{by a property of the direct image} \\
&= H^*(Z, \mathbb{D}_Z) \\
&= H_*(Z)
\end{aligned}$$

Keeping track of the indices and gradings in the above computation, one obtains that $H(Z) = \mathrm{End}_{D^b(N)}(\mathcal{L})$. The compatibility of the algebra structures is a consequence of base change; it is helpful to consider the diagram

$$\begin{tikzcd}
& & M \times_N M \times_N M \arrow[dll] \arrow[d] \arrow[drr] & & \\
Z \arrow[d] \arrow[drr] & & Z \arrow[dll] \arrow[drr] & & Z \arrow[dll] \arrow[d] \\
M \arrow[drr] & & M \arrow[d] & & M \arrow[dll] \\
& & N & &
\end{tikzcd} .$$

∎

3 An application of BBD Decomposition

In this section we see how the Springer correspondence from last time follows from a decomposition theorem of Beilinson, Bernstein, and Deligne. We will not state the theorem in full generality, only the special cse that is relevant for us.

3.1 Semi-smallness

We adopt the setting of the previous section. Assume that the smooth complex variety M is connected and that the proper map $\mu : M \to N$ is projective. For $x \in N$, let $M_x = \mu^{-1}(x)$ denote the the fiber over x.

It is possible to find a finite stratification $N = \coprod_\alpha N_\alpha$ of N into locally closed smooth subvarieties N_α such that $\mu : \mu^{-1}(N_\alpha) \to N_\alpha$ is a locally trivial topological fibration.

Definition. The map μ is **semi-small** if for every stratum N_α and every $x \in N_\alpha$, we have

$$\dim_{\mathbb{C}}(N_\alpha) - 2\dim_{\mathbb{C}}(M_x) = \dim_{\mathbb{C}}(M)$$

Remark. This definition can be interpreted as saying the the fibers of μ don't 'grow very quickly'. Some authors would refer to the above condition as *strictly* semi-small and would reserve the term semi-small for the condition where we replace the equals sign by '$\leq$'.

Theorem 2. *The Springer resolution is semi-small.*

3.2 Intersection cohomology complexes

We adopt the following notation. Let $\phi = (N_\phi, \chi_\phi)$ denote a pair consisting of a stratum N_ϕ of N (that is, $N_\phi = N_\alpha$ for some α) and an irreducible local system $\chi_\phi \in \mathsf{LS}(N_\phi)$ on the stratum N_ϕ. Two such pairs $\phi = (N_\phi, \chi_\phi)$ and $\psi = (N_\psi, \chi_\psi)$ are called isomorphic if $N_\phi = N_\psi$ and, in addition, the local systems χ_ϕ and χ_ψ are isomorphic.

There is a procedure due to Deligne-Goresky-MacPherson for associating a object IC_ϕ of $D^b(N)$ to a pair $\phi = (N_\phi, \chi_\phi)$, called the **intersection cohomology complex** of ϕ. Among the properties of the intersection cohomology complexes are the following:

- Non-isomorphic pairs ϕ and ψ give rise to non-isomorphic intersection cohomology complexes.
- The category $\mathsf{Perv}(N)$ of perverse sheaves on N can be defined as a certain heart of the triangulated category $D^b(N)$ of constructible complexes. In particular, $\mathsf{Perv}(N)$ is a full abelian subcategory of $D^b(N)$. The intersection cohomology sheaves IC_ϕ are objects of $\mathsf{Perv}(N)$.
- Every IC_ϕ is a simple object of $\mathsf{Perv}(N)$, and we have

  $$\mathrm{Hom}_{\mathsf{Perv}(N)}(IC_\phi, IC_\psi) = \mathbb{C}\delta_{\phi,\psi}.$$

  Note that the same equation holds if we replace $\mathsf{Perv}(N)$ by $D^b(N)$.

3.3 Special case of BBD

We are in a position to state a special case of the Beilinson-Bernstein-Deligne decomposition theorem:

Theorem 3. *Let $\mu : M \to N$ as above. If μ is semi-small, then*

1. $\mu_*(\mathbb{C}_M[m]) = \bigoplus_{\phi=(N_\phi,\chi_\phi)} L_\phi \otimes IC_\phi$ *for some finite-dimensional vector spaces* L_ϕ.
2. *For any α, the collection $\{H(M_x) \mid x \in N_\alpha\}$ forms a local system on N_α.*
3. *The vector space L_ϕ can be identified with* $\mathrm{Hom}_{\pi(N,x)}(H(M_x), \chi_\phi)$.

A possibly more precise way to state the second part of the theorem is that there exists a local system on N_α whose stalk at x is precisely the homology of the fiber M_x. This is a consequence of the assumption that μ is a locally trivial fibration over each stratum.

In the last statment, we use the fact that the category of local systems on a (smooth) stratum N_α is equivalent to the category of representations of the fundamental group $\pi(N_\alpha, x)$ of N_α at any point x.

We conclude with the following computation:

$$\begin{aligned}
H(Z) &= \mathrm{End}_{\mathsf{Perv}(N)}(\mathcal{L}, \mathcal{L}) \\
&= \mathrm{Hom}_{\mathsf{Perv}(N)}(\mu_*(\mathbb{C}_M[m]), \mu_*(\mathbb{C}_M[m])) \\
&= \mathrm{Hom}_{\mathsf{Perv}(N)}(\bigoplus_\phi L_\phi \otimes IC_\phi, \bigoplus_\psi L_\psi \otimes IC_\psi) \\
&= \bigoplus_\phi \mathrm{Hom}_{\mathsf{Perv}(N)}(L_\phi \otimes IC_\phi, \bigoplus_\psi L_\psi \otimes IC_\psi) \\
&= \bigoplus_\phi \mathrm{Hom}_{\mathsf{Perv}(N)}(L_\phi \otimes IC_\phi, L_\phi \otimes IC_\phi) \\
&= \bigoplus_\phi \mathrm{Hom}_{\mathbb{C}}(L_\phi, L_\phi) \\
&= \bigoplus_\phi \mathrm{End}_{\mathbb{C}}(L_\phi)
\end{aligned}$$

Therefore, the irreducible representations of $H(Z)$ can be identified with the nonzero elements of the collection $\{L_\phi = \mathrm{Hom}_{\pi(N,x)}(H(M_x), \chi_\phi)\}$.

Note: In the fourth equality of the above computation, the coproduct in the first variable of the hom comes out as a coproduct becuase (1) we assume that the stratification is finite, (2) only finitely many of the L_ϕ with fixed N_ϕ are nonzero since only finitely many irreducbile representations of $\pi_1(N, x)$ can appear in a finite-dimensional vector space, and (3) $\mathsf{Perv}(N)$ is an abelian category, so finite products and coproducts coincide.

References

[1] N. Chriss and V. Ginzburg. Representation theory and complex geometry. Birkhäuser, 1997.