Quantum $\mathfrak{sl}_2$

(Lecture notes summary)

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1. Introduction

In this talk, we construct the Drinfeld-Jimbo quantization of the universal enveloping algebra of $\mathfrak{sl}_2$. This quantization emerges from consideration of the Sklyanin Poisson-Lie bracket on the group SL_2, and corresponding additional structure on the Lie algebra $\mathfrak{sl}_2$. There is a menagerie of Hopf algebras that bear the name 'quantum group'; we define each of them in the case of $\mathfrak{sl}_2$ and illustrate their interrelations. These Hopf algebras include the formal deformation, the rational form, the Lusztig integral form, the De Concini - Kac integral form, and the small quantum group. We give an overview of the representation theory and structure of several of these algebras, and describe phenomena that emerge when the quantum parameter is a root of unity.

Before specializing to the case of $\mathfrak{sl}_2$, however, we begin with an overview of the general principles behind the theory of quantum groups.

2. The classical picture

The main objects are:

- Start with a semisimple linear algebraic group G over $\mathbb{C}$.
- Let $\mathfrak{g}$ denote the Lie algebra of G.
- Let $U\mathfrak{g}$ be the universal enveloping algebra. This is a Hopf algebra.

We will consider the following extra structures on these objects, which will be defined more precisely later on:

- Poisson-Lie structures on the group G. The motivation for these structures originates in classical mechanics for a dynamical system whose phase space is a group.
- Lie bialgebra structures on the Lie algebra $\mathfrak{g}$. These structures are obtained through differentiation of the Poisson bracket on G.
- Co-Poisson-Hopf algebra structures on the Hopf algebra $U\mathfrak{g}$. These structures are obtained by extending the Lie bialgebra structure on $\mathfrak{g}$.

More specifically, we will be interested in a single example of each of these structures:

- The Sklyanin Poisson-Lie bracket on G, which originates in classical inverse scattering theory.
- The 'standard' Lie bialgebra structure on $\mathfrak{g}$, which is obtained from differentiating the Sklyanin bracket. It depends on a choice of Cartan subalgebra, and can be characterized in terms of its restriction to $\mathfrak{sl}_2$-triples.
- The corresponding 'standard' co-Poisson-Hopf algebra structure on $U\mathfrak{g}$, denoted δ_{std}.

3. Quantization

The passage from classical mechanics to quantum mechanics corresponds to the quantization of the universal enveloping algebra equipped with its standard co-Poisson-Hopf algebra structure.

$$(U\mathfrak{g}, \delta_{\mathrm{std}}) \underset{\text{classical limit } \hbar=0}{\overset{\text{quantization}}{\rightleftarrows}} U_\hbar\mathfrak{g}$$

Specifically, we construct a Hopf algebra $U_\hbar\mathfrak{g}$ over the ring $\mathbb{C}[[\hbar]]$ of formal power series in the variable $\hbar$ whose evaluation at $\hbar = 0$ recovers $U\mathfrak{g}$ with its Hopf algebra structure, and the Lie bialgebra structure appears through cocommutators. In other words, $(U\mathfrak{g}, \delta_{\mathrm{std}})$ is the classical limit of $U_\hbar\mathfrak{g}$. The algebra $U_\hbar\mathfrak{g}$ is called the formal deformation as it defined over the algebra $\mathbb{C}[[\hbar]]$ of functions on the formal disk.

One can further consider a 'rational form' of the quantum group as a Hopf algebra $U_{\mathbb{Q}(t)}(\mathfrak{g})$ defined over the field $\mathbb{Q}(t)$ of rational functions in the variable t. For any transcendental complex number q there is an evaluation morphism $\mathbb{Q}(t) \to \mathbb{C}$, and we can specialize $U_{\mathbb{Q}(t)}(\mathfrak{g})$ along this map to obtain a finitely generated Hopf algebra over $\mathbb{C}$. The rational form includes into the formal deformation (with scalars extended to the fraction field of $\mathbb{C}[[\hbar]]$) with $t \mapsto e^\hbar$.

In order to evaluate to non-transcendental complex numbers, one must choose an integral form of the quantum group, namely, a $\mathcal{A} = \mathbb{Z}[t, t^{-1}]$-subalgebra of the rational form $U_{\mathbb{Q}(t)}(\mathfrak{g})$ whose extension of scalars to $\mathbb{Q}(t)$ recovers the full rational form. There are two main choices for this integral form, known as the De Concini - Kac form $U_\mathcal{A}^{\mathrm{DK}}(\mathfrak{g})$ and the Lusztig $U_\mathcal{A}^{\mathrm{Lus}}(\mathfrak{g})$. These can be evaluated to any nonzero complex number $q \in \mathbb{C}^\times$, and the two specializations coincide when q is not a root of unity. (If q is transcendental, they also coincide with the specialization of the rational form.) We denote this common specialization by $U_q\mathfrak{g}$.

If q is a root of unity, however, the specializations $U_q^{\mathrm{DK}}(\mathfrak{g})$ and $U_q^{\mathrm{Lus}}(\mathfrak{g})$ differ drastically in their structure and representation theory. We will see that the category of finite-dimensional modules for the Lusztig form $U_q^{\mathrm{Lus}}(\mathfrak{g})$, although not semisimple, has irreducible objects indexed by dominant weights, just like the classical enveloping algebra $U\mathfrak{g}$. Moreover, the two algebras are linked via Lusztig's quantum Frobenius morphism

$$\mathrm{Fr} : U_q^{\mathrm{Lus}}(\mathfrak{g}) \longrightarrow U\mathfrak{g}.$$

This map of Hopf algebras is surjective, and its kernel is generated by the augmentation ideal of a finite-dimensional sub-Hopf algebra $u_q\mathfrak{g}$ of $U_q^{\mathrm{Lus}}(\mathfrak{g})$, known as the small quantum group.

On the other hand, the DeConcini-Kac specialization at a root of unity q contains a large central subalgebra isomorphic to the algebra of functions $\mathcal{O}(G^*)$ of the dual Poisson-Lie group G^* of G, and its representation theory can be understood through this subalgebra and the full center $Z_q = Z(U_q^{\mathrm{DK}}(\mathfrak{g}))$. In fact, the dimensions of the irreducible representations are bounded, and irreducibles are generically determined by their central character, valued in $\mathrm{Spec}(Z_q)$, which is finite over G^*. Finally, the quotient of $U_q^{\mathrm{DK}}(\mathfrak{g})$ by the augmentation ideal of $\mathcal{O}(G^*)$ recovers the small quantum group, and thus there is a map:

$$U_q^{\mathrm{DK}}(\mathfrak{g}) \twoheadrightarrow u_q\mathfrak{g} \hookrightarrow U_q^{\mathrm{Lus}}(\mathfrak{g})$$

4. Cheat sheet

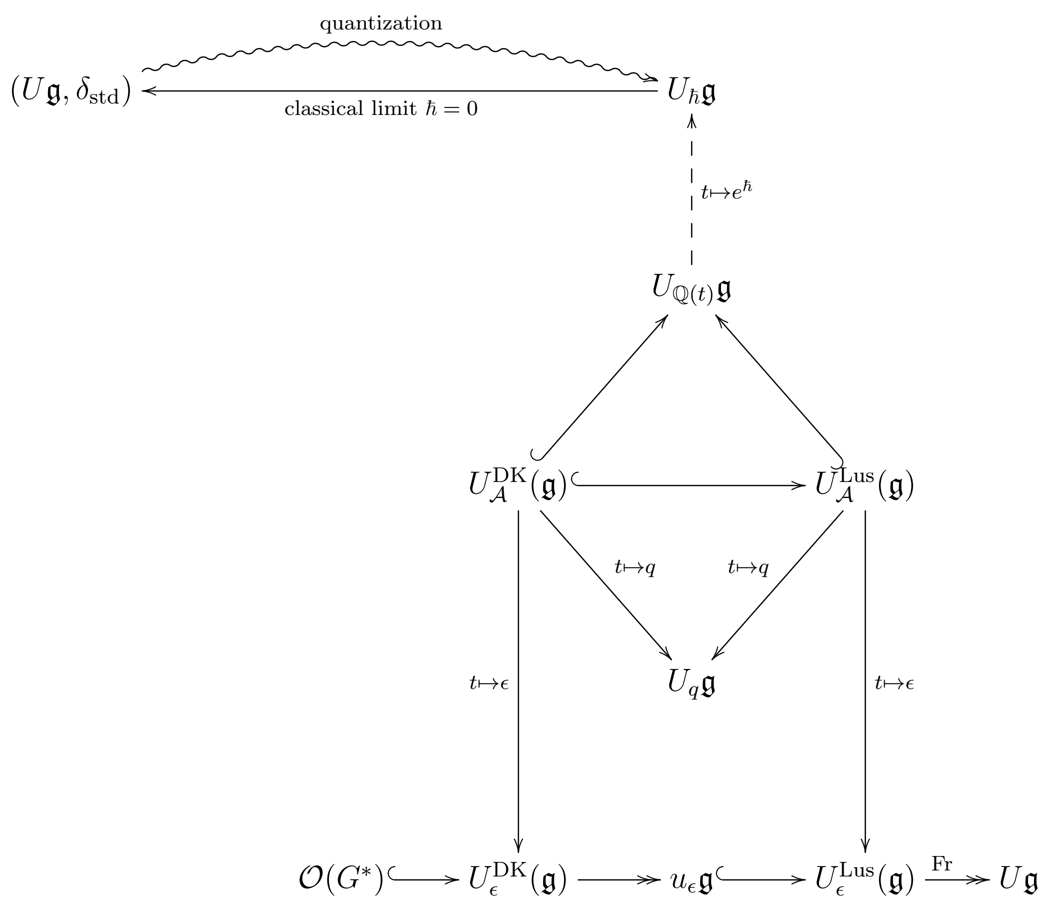
To summarize the content of the previous section:

- The formal deformation $U_\hbar\mathfrak{g}$ lives over the formal disk.
- The rational form $U_{\mathbb{Q}(t)}(\mathfrak{g})$ lives over the transcendental numbers of $\mathbb{C}$.
- The integral forms $U_{\mathcal{A}}^{\mathrm{DK}}(\mathfrak{g})$ and $U_{\mathcal{A}}^{\mathrm{Lus}}(\mathfrak{g})$ live over the complex numbers $\mathbb{C}$.
- For q not a root of unity, the specializations $U_q^{\mathrm{DK}}(\mathfrak{g})$ and $U_q^{\mathrm{Lus}}(\mathfrak{g})$ coincide, giving $U_q\mathfrak{g}$.
- For q a root of unity, we have the morphisms:

$$\mathcal{O}(G^*) \hookrightarrow U_q^{\mathrm{DK}}(\mathfrak{g}) \twoheadrightarrow u_q\mathfrak{g} \hookrightarrow U_q^{\mathrm{Lus}}(\mathfrak{g}) \overset{\mathrm{Fr}}{\twoheadrightarrow} U\mathfrak{g}$$

  where the first two maps form a 'short exact sequence of Hopf algebras', as do the last two maps.

These various versions of the quantum group fit into the following diagram, in which, for clarity, we denote by q a fixed nonzero complex number that is not a root of unity, and by ϵ a primitive ℓ-th root of unity ($\ell > 1$ odd).



5. Representation theory

When q is generic, there is an equivalence of monoidal categories between (type $\mathbf{1}$) finite-dimensional representations of $U_q\mathfrak{g}$ and finite-dimensional representations of the classical enveloping algebra $U\mathfrak{g}$. In particular, both categories are semisimple with irreducible objects indexed by dominant integral weights. However, the two categories differ in their braidings.

When q is a primitive ℓ-th root of unity ($\ell > 1$ odd), the finite-dimensional irreducible representations of the Lusztig form $U_\epsilon^{\mathrm{Lus}}(\mathfrak{g})$ are also in bijection with the dominant integral weights, and these can be understood in terms of the quantum Frobenius map. In particular, the finite-dimensional irreducible representations of $U\mathfrak{g}$ correspond to linear combinations of the fundamental weights all of whose coefficients are divisible by ℓ. However, the category of finite-dimensional representations is not semisimple It does, however, carry the structure of a braided monoidal category.

When q is a primitive ℓ-th root of unity ($\ell > 1$ odd), the irreducible representations of the small quantum group $u_q\mathfrak{g}$ are in bijection a finite subset of dominant weights, namely, those that are non-negative linear combinations of the fundamental weights with coefficients less than ℓ. All of these extend to give a representation of the Lusztig form, compatibly with the indexing in terms of weights. Just like with the Lusztig form, the category of finite-dimensional representations of the small quantum group is not semisimple, but does carry the natural structure of a braided monoidal category.

Let q be a primitive ℓ-th root of unity ($\ell > 1$ odd), and let Z denote the center of the De Concini - Kac form $U_\epsilon^{\mathrm{DK}}(\mathfrak{g})$. Each finite-dimensional irreducible representation of $U_\epsilon^{\mathrm{DK}}(\mathfrak{g})$ has a central character, which defines a point in $\mathrm{Spec}(Z)$. There is a Zariski closed subset $D \subseteq \mathrm{Spec}(Z)$ such that

- For any point χ in the complement $\mathrm{Spec}(Z) \setminus D$, there is a unique irreducible representation with central character χ. In fact, this is the Azumaya locus of $U_\epsilon^{\mathrm{DK}}(\mathfrak{g})$ over $\mathrm{Spec}(Z)$.
- For any point χ in D, there are finitely many irreducible representations with central character χ.

This picture can be understood further in terms of the Poisson-Lie dual group G^* and the inclusion $\mathcal{O}(G^*) \hookrightarrow Z$. While the category of finite-dimensional representations of the De Concini - Kac form is monoidal, it is not braided monoidal.

6. Presentations for $\mathfrak{sl}_2$

In this section, we give presentations of the various algebras mentioned above in the case of $\mathfrak{sl}_2$. Fix $T \subseteq \mathrm{SL}_2$ to be the maximal torus of diagonal matrices, and $B \subseteq \mathrm{SL}_2$ the Borel subgroup of upper-triangular matrices. The opposite Borel B^- consists of the lower-triangular matrices, and we have that $B \cap B^- = T$. There is a unique (up to scaling) non-trivial Poisson-Lie structure on SL_2 that restricts to Poisson-Lie structures on B and B^- that agree on T, and it is given by:

Definition 6.1. The Sklyanin Poisson-Lie bracket on $\mathcal{O}(\mathrm{SL}_2) = \mathbb{C}[a,b,c,d]/(ad - bc = 1)$ is given by:

$$\{a,b\} = ab, \quad \{a,c\} = ac, \quad \{a,d\} = 2bc, \quad \{b,c\} = 0, \quad \{b,d\} = bd, \quad \{c,d\} = cd.$$

The Lie algebra $\mathfrak{sl}_2$ generated over $\mathbb{C}$ by elements E, F, H with Lie bracket defined by $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. Differentiating the Sklyanin bracket on $\mathcal{O}(\mathrm{SL}_2)$, we obtain the so-called 'standard' Lie bialgebra structure on $\mathfrak{sl}_2$:

Definition 6.2. The standard Lie bialgebra structure on $\mathfrak{sl}_2$ is given by

$$\delta : \mathfrak{sl}_2 \to \mathfrak{sl}_2 \otimes \mathfrak{sl}_2$$

$$E \mapsto E \wedge H, \qquad F \mapsto F \wedge H, \qquad H \mapsto 0.$$

The universal enveloping algebra $U\mathfrak{sl}_2$ is generated as an algebra over $\mathbb{C}$ by elements E, F, H with relations given by the commutators:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = H.$$

In addition, $U\mathfrak{sl}_2$ has a Hopf algebra structure given extending

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \qquad \epsilon(X) = 0, \qquad S(X) = -X$$

for $X \in \{E, F, H\}$. Note that $U\mathfrak{sl}_2$ is cocommutative as a Hopf algebra, since $\Delta(a) = \Delta^{\mathrm{op}}(a)$ for any $a \in U\mathfrak{sl}_2$.

Definition 6.3. The 'standard' co-Poisson Hopf structure on $U\mathfrak{sl}_2$ is defined by extending the map δ from above to a map $U\mathfrak{sl}_2 \to U\mathfrak{sl}_2 \otimes U\mathfrak{sl}_2$ using the condition

$$\delta(a_1 a_2) = \delta(a_1)\Delta(a_2) + \Delta(a_1)\delta(a_2)$$

for any a_1 and a_2 in $U\mathfrak{sl}_2$.

The 1-cocycle condition on δ will guarantee that $\delta(a_1 a_2 - a_2 a_1) = \delta([a_1, a_2])$ for elements $a_1, a_2 \in \mathfrak{sl}_2$.

Definition 6.4. The algebra $U_\hbar(\mathfrak{sl}_2)$ is the quotient of the free algebra over $\mathbb{C}[[\hbar]]$ on elements E, F, and H by the topological closure in the $\hbar$-adic topology of the ideal generated by the elements:

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.$$

The Hopf structure is given by:

$$\Delta_\hbar(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta_\hbar(E) = E \otimes 1 + e^{\hbar H} \otimes E, \qquad \Delta_\hbar(F) = F \otimes e^{-\hbar H} + 1 \otimes F$$

$$S(H) = -H, \qquad S(E) = -e^{-\hbar H} E, \qquad S(F) = -F e^{\hbar H}$$

$$\epsilon(E) = \epsilon(F) = \epsilon(H) = 0$$

The algebra $U_\hbar(\mathfrak{sl}_2)$ quantizes the standard co-Poisson Hopf structure on $U\mathfrak{sl}_2$. More precisely:

Lemma 6.5. *The following two conditions are satisfied:*

(1) We recover the Hopf algebra $U\mathfrak{sl}_2$ from $U_\hbar(\mathfrak{sl}_2)$ by setting $\hbar = 0$.
(2) The following identity holds for $X \in \{E, F, H\}$:

$$\delta(X) \equiv \frac{\Delta_\hbar(X) - \Delta_\hbar^{\mathrm{op}}(X)}{\hbar} \mod \hbar$$

By abuse of notation, in the last equation we think of X on the left-hand-side as an element of the Lie algebra $\mathfrak{sl}_2$, and as an element of $U_\hbar(\mathfrak{sl}_2)$ on the right-hand-side. Note that the right-hand-side is well-defined because $U\mathfrak{sl}_2$ is cocommutative. The algebra $U_\hbar(\mathfrak{sl}_2)$ is the unique quantization (up to automorphisms of $\mathbb{C}[[\hbar]]$ of the form $\hbar \mapsto \hbar + \mathcal{O}(\hbar^2)$) that has an analogue of the Cartan involution

$$E \mapsto F, \qquad F \mapsto E, \qquad H \mapsto -H.$$

Definition 6.6. The rational form $U_{\mathbb{Q}(t)}(\mathfrak{sl}_2)$ of the quantum group for $\mathfrak{sl}_2$ is generated over $\mathbb{Q}(t)$ by generators E, F, and K with relations:

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

The Hopf structure is given by:

$$\Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes 1 + K \otimes E, \qquad \Delta(F) = F \otimes K^{-1} + 1 \otimes F,$$
$$S(K) = K^{-1}, \qquad S(E) = -K^{-1}E, \qquad S(F) = -FK,$$
$$\epsilon(E) = \epsilon(F) = 0, \qquad \epsilon(K) = 1.$$

These relations are obtained from the formal deformation $U_\hbar(\mathfrak{sl}_2)$ by setting $K = e^{\hbar H}$ and $t = e^{\hbar}$. We now define the De Concini - Kac and Lusztig integral forms, which are subalgebras of the rational form $U_{\mathbb{Q}(t)}(\mathfrak{sl}_2)$, defined over the the subring $\mathcal{A} = \mathbb{Z}[t, t^{-1}]$ of $\mathbb{Q}(t)$. In fact, they are sub-Hopf algebras.

Definition 6.7. Define the following subalgebras of the rational form $U_{\mathbb{Q}(t)}(\mathfrak{sl}_2)$:

- The De Concini - Kac integral form $U_{\mathcal{A}}^{\mathrm{DK}}(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra generated by the elements
$$E,\ F,\ K,\ K^{-1},\ \text{and } L := \frac{K - K^{-1}}{t - t^{-1}}$$
- The Lusztig form $U_{\mathcal{A}}^{\mathrm{Lus}}(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra generated by the elements
$$\frac{E^r}{[r]_q!},\ \frac{F^r}{[r]_q!},\ K,\ \text{and } K^{-1}$$
for $r = 1, 2, 3, \ldots$.

Here $[r]_q = \frac{q^r - q^{-r}}{q - q^{-1}}$ is the quantum integer, and $[r]_q! = [r]_q[r-1]_q \cdots [2]_q[1]_q$ it the quantum factorial. The elements $\frac{E^r}{[r]_q!}$ and $\frac{F^r}{[r]_q!}$ are referred to as the 'divided powers' of E and F, and the Lusztig form is also known as the 'restricted' form.