Unipotent characters of finite groups of Lie type

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Abstract

These notes were written in preparation for a talk given by the author in UT Austin's graduate student geometry seminar. Finite groups of Lie type arise as Frobenius fixed points of certain linear algebraic groups, and the first part of the talk will serve as an introduction to such finite groups. The second part of the talk will explore the representation theory of these groups; we will focus on the unipotent characters and their relationship to the Weyl group of the original linear algebraic group. Finally, we give an indication of how the representation theory of finite groups of Lie type may be re-interpreted using topological field theory.

1 Introduction: finite groups of Lie type

Let $K = \bar{\mathbb{F}}_p$ and consider the linear algebraic group $\mathrm{GL}_n K$. For every $q = p^r$, we have a Frobenius map

$$\sigma_q : \mathrm{GL}_n K \to \mathrm{GL}_n K$$

that raises every entry of a matrix to the q-th power: $(a_{ij}) \mapsto (a_{ij}^q)$. The map σ_q is a bijective homomorphism of algebraic groups, but its inverse is not a morphism of varieties. Observe that, by basic Galois theory, the set of fixed points of σ_q is the finite group $\mathrm{GL}_n\mathbb{F}_q$.

The finite group $\mathrm{GL}_n\mathbb{F}_q$ is our first example of a finite group of Lie type. Momentarily I will give the general construction of all finite groups of Lie type, but the rough idea is that they are groups of matricies with entries in a finite field. Crucially, we regard the result as the fixed points of an endomorphism of a linear algebraic group over an algebraically closed field. The latter groups have rich structure that is relatively well-understood; we can access this structure in order to understand the resulting finite groups.

We now give the general construction of finite groups of Lie type. Let G be a connected reductive group over $K = \bar{\mathbb{F}}_p$. A **Frobenius map** on G is a homomorphism of algebraic groups $F : G \to G$ such that there is an embedding $G \hookrightarrow \mathrm{GL}_n K$ for some n and $q = p^r$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{GL}_n K \arrow[r, "\sigma_q"] & \mathrm{GL}_n K \\ G \arrow[u] \arrow[r, "F^i"] & G \arrow[u] \end{tikzcd}$$

In other words, the restriction of the 'standard' Frobenius map σ_q to G equals the i-fold composition $F \circ F \circ \cdots \circ F$. Observe that the fixed points G^F form a finite group. A group that arises in this way is called a **finite group of Lie type**.

Remark. Under some hypotheses, if $F : G \to G$ is a homomorphism such that G^F is finite, then F is a Frobenius map.

Before giving some examples, let us recall some facts about simple groups and Dynkin diagrams. Suppose G is a simple linear algebraic group over K. Then G is determined by two pieces of data:

- its Dynkin diagram.
- its isogeny type (e.g. simply connected, adjoint, or in between).

I assume most of the audience is familiar with Dynkin diagrams. For those not familiar with isogeny types, consider it as an extra piece of combinatorial data. At one extreme is the simply connected type and at the other extreme is the adjoint type.

Below are some examples of the sorts of finite groups that arise from certain simple groups. The Dynkin diagram and isogeny type determine a simple group, and various choices of Frobenius maps give rise to the specfied finite groups. This is far from an exhaustive list.

- Type A_l

  simply connected: $\mathrm{SL}_{l+1}(q)$ or $\mathrm{SU}_{l+1}(q^2)$

  adjoint: $\mathrm{PSL}_{l+1}(q)$
- Type B_l

  simply connected: $\mathrm{Spin}_{2l+1}(q)$

  adjoint: $\mathrm{SO}_{2l+1}(q)$
- Type C_l

  simply connected: $\mathrm{Sp}_{2l}(q)$
- Type D_l

  simply connected: $\mathrm{Spin}_{2l}(q)$

  others: $\mathrm{SO}_{2l}(q)$

If G is simple and simply connected, then the quotient $G^F/Z(G^F)$ is a almost always an (abstract) simple group. If G is simple and adjoint, then the derived subgrgoup $[G^F, G^F]$ is almost always an (abstract) simple group. Thus finite groups of Lie type are a source of fintie simple groups. In fact, they feature prominently in the **classification of finite simple groups**:

1. The cyclic groups of prime order $\mathbb{Z}/p\mathbb{Z}$.
2. The alternating groups A_n for $n \geq 5$.
3. Simple finite groups of Lie type.
4. 26 sporadic groups.

What can we say about the representation theory of these groups? The case (1) is easy, and the case (2) is closely related to the representation theory of the symmetric group, which has been understood for a long time. The character tables of the 26 sporadic groups have been computed.

This leaves the fintie groups of Lie type, whose representation theory is more difficult, and studied extensively by Deligne and Lusztig using techniques of algebraic geometry. The irreducible characters of these groups have been understood in some sense, but many of the constructions are mysterious and seemingly ad hoc. There is some hope that topological field theory will provide re-interpretations and cleaner results in the subject; this is currently work in progress.

In the next part of the talk, we give a flavor of Lusztig's apporach to studying one aspect of the representation theory of finite groups of Lie type, namely the unipotent characters.

2 The characters R_w and the unipotent characters

Let G and F be as above and let W be the Weyl group of G. The idea of unipotent characters is to connect the representation theory of W to the representation theory of G^F. In particular, we will define a map $\mathrm{Cl}(W) \to \mathrm{Cl}(G^F)$ from class functions on W to class functions on G^F. Under some hypotheses, this map will respect the usual inner product on class functions for a finite group. In order to obtain this goal, we need to relate G to W using the flag variety of G. First we recall some standard facts.

Let B be a Borel subgroup of G. Then the **flag variety** of G is defined as $\mathcal{B} = G/B$. One can also define the flag variety as the set of Borel subgroups of G. To see that the two definitions are equivalent, recall that G acts transitively on the set of Borel subgroups by conjugation and the normalizer of a Borel subgroup is itself.

Observe that G acts on $\mathcal{B}$ on the left, and diagonally on the product $\mathcal{B} \times \mathcal{B}$. The orbits can be identified with the double coset space $B\backslash G/B$. The **Bruhat decomposition** implies that $B\backslash G/B$ can be identified with W. In summary:

$$G\backslash(\mathcal{B} \times \mathcal{B}) = B\backslash G/B = W.$$

Let $\mathcal{O}_w \subset \mathcal{B} \times \mathcal{B}$ be the orbit corresponding to $w \in W$. We say that a pair (B_1, B_2) of Borel subgroups are in **relative position** w if $(B_1, B_2) \in \mathcal{O}_w$.

Now assume that the chosen Borel B is F-stable, that is, $F(B) = B$. For each $w \in W$, define a variety

$$\mathcal{B}_w = \{B' \in \mathcal{B} \mid (B', F(B')) \in \mathcal{O}_w\}.$$

The cartoon here is that we draw $\mathcal{B} \times \mathcal{B}$ as a square, divide it up into orbits for the diagonal action of G, and then draw the graph of F. Then $\mathcal{B}_w$ consists of the Borels for which the corresponding point on the graph lies in $\mathcal{O}_w$.

One can verify that the finite group G^F acts on each variety $\mathcal{B}_w$. The action is by left translation if we regard $\mathcal{B}$ as the quotient G/B, or by conjugation if we regard $\mathcal{B}$ as the set of all Borel subgrgoups. Therefore, G^F acts on the cohomomlogy on $\mathcal{B}_w$. In this case, the correct cohomology theory to consider is compactly-supported ℓ-adic cohomology, where ℓ is a prime different from p:

$$G^F \curvearrowright H^i_c(\mathcal{B}_w, \bar{\mathbb{Q}}_\ell).$$

One justification for using ℓ-adic cohomology is that it interacts nicely with varieties over fields of positive characteristic. Thus, we obtain a representation of G^F for each $i \geq 0$. However, we will actually be interested in the alternating sum of the traces of these representations, and define a generalized character of G^F as

$$R_w(g) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(g, H^i_c(\mathcal{B}_w, \bar{\mathbb{Q}}_\ell)).$$

It turns out that the character R_w depends only on the conjugacy class of W. These characters play an important role is describing the characters of G^F.

A character χ of G^F is called **unipotent** if $\langle R_w, \chi \rangle \neq 0$ for some $w \in W$. In other words, the unipotent characters of G^F are those irreducible characters that appear as constituents of the characters R_w for various $w \in W$. We adopt the notation $(\hat{G}^F)_u$ for the set of unipotent characters of G^F. We will be intereseted in expressing the R_w in terms of the unipotent characters.

Define a map

$$\begin{aligned} A : \mathrm{Cl}(W) &\to \mathrm{Cl}(G^F) \\ \delta_w &\mapsto R_w \end{aligned}$$

where δ_w is the delta function on the conjugacy class of w in W scaled by the size of the centralizer; explicitly, $\delta_w(y) = |C_W(w)|$ if w and y are conjugate and 0 otherwise.

Recall that class functions on a finite group have a usual inner product, and we can ask if the map A preserves this inner product. This is true if we assume that G^F is **split**, which will be an assumption that we will adopt for the remainder of this talk. (We prefer not to define this notion precisely here. The idea that F induces an automorphism of the Weyl group W of G, and G^F is split when this automorphism is the identity.) To reiterate, if G^F is split, then the map A preserves inner products; this is a nontrivial fact that relies on an inner product formula due to Lusztig.

Remark. There should be an interpretation of this map A as pulling and pushing charachter sheaves along the so-called horocycle correspondence

$$B\backslash G/B \leftarrow \frac{G}{B} \rightarrow \frac{G}{G}$$

and using Grothendieck's sheaf-function correspondence. Recall that $B\backslash G/B$ can be identified with W, and the Frobenius map acts trivially on W, so $W^F = W$. I'm sure that this is something well-known, but I haven't read up on that yet and can't give you the precise formulation at the moment.

The following exercise addresses the question of where the map A sends the irreducible characters of W:

Exercise. Let $\hat{W}$ denote the set of irreducible characters of W.

1. The image of an irreducible character ϕ under A is given by

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w.$$

   In particular, each R_ϕ is a generalized character of G^F with norm 1.

2. Also, we can write the R_w in terms of the R_ϕ:

$$R_w = \sum_{\phi \in \hat{W}} \phi(w) R_\phi. \tag{1}$$

Since the R_ϕ are generalized characters of norm 1 and are related to the irreducible characters of W, there is an expectiation that the R_ϕ are themselves irreducible or at least 'close to' irreducible characters. It turns out that in type A, the R_ϕ are precisely the unipotent characters. This is not true in other types; still, we adopt Lusztig's nomenclature and refer to the R_ϕ as the **almost characters** of G^F.

3 Lusztig's non-abelian Fourier transform

In this section, we present the precise relationship (due to Lusztig) that relates the unipotent characters to the characters R_w. We begin with a few facts, whose explanation is suppressed:

- The unipotent characters $(\hat{G}^F)_u$ are divided into families: $(\hat{G}^F)_u = \coprod_i \mathcal{F}_i$.
- Using the map A, there is a corresponding division of the irreducible characters $\hat{W}$ of W into families: $\hat{W} = \coprod_i \hat{W}(\mathcal{F}_i)$.
- There is a bijection between the families in $(\hat{G}^F)_u$ and the families in $\hat{W}$.

Remark. In fact, there are bijections

$$\left\{\begin{matrix}\text{families}\\ \text{in } (\hat{G}^F)_u\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{families}\\ \text{in } \hat{W}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{2-sided cells}\\ \text{in } W\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unipotent conjugacy}\\ \text{classes in } {}^LG\end{matrix}\right\}.$$

where ${}^LG$ denotes the Langlands dual group of G. In this way, the study of unipotent characters connects to Hecke algebras, Kazhdan-Lusztig polynomials, Langlands duality, and many other topics in geometric representation theory.

Lusztig had the insight to assign to each family $\mathcal{F} \subset (\hat{G}^F)_u$ a finite group $\Gamma(\mathcal{F})$ with

$$\Gamma(\mathcal{F}) \in \{1, (\mathbb{Z}/2\mathbb{Z})^e, S_3, S_4, S_5\}.$$

Briefly, the way it works is that each family $\mathcal{F}$, has a so-called 'special character' $\psi \in \hat{W}(\mathcal{F})$. There is a polynomial in $\mathbb{Q}[t]$ associated to ψ, and the smallest power of t appearing in this polynomial has coefficient given by $1/|\Gamma(\mathcal{F})$, for some finite group $\Gamma(\mathcal{F}) \in \{1, (\mathbb{Z}_2)^e, S_3, S_4, S_5\}$, where e is a positive integer.

Notation: For a finite group Γ, the $M(\Gamma)$ is defined as the set of all pairs (x, σ) where x is a conjugacy class representative and σ is an irreducible character of the centralizer $C_\Gamma(x)$. In the case that $\Gamma = \Gamma(\mathcal{F})$, we abbreviate the set $M(\Gamma(\mathcal{F}))$ by $M(\mathcal{F})$.

Lusztig proves that:

- There is a bijection $M(\mathcal{F}) \to \mathcal{F}$. We denote the unipotent character in $\mathcal{F}$ corresponding to the pair $(x, \sigma) \in M(\mathcal{F})$ by $\chi^{\mathcal{F}}_{(x,\sigma)}$.
- There is an injection $\hat{W}(\mathcal{F}) \to M(F)$. We denote the image of $\phi \in \hat{W}(\mathcal{F})$ by $(x_\phi, \sigma_\phi) \in M(F)$.
- For $\phi \in \hat{W}(\mathcal{F})$, we have the following formula

  $$\langle \chi^{\mathcal{F}}_{(x,\sigma)}, R_\phi \rangle = \frac{\pm 1}{|C_{\Gamma(\mathcal{F})}(x)||C_{\Gamma(\mathcal{F})}(x_\phi)|} \sum_{\substack{g \in \Gamma(\mathcal{F}) \\ gxg^{-1} \in C_{\Gamma(\mathcal{F})}(x_\phi)}} \sigma(g^{-1} x_\phi g) \sigma_\phi(g x^{-1} g^{-1}).$$

  that gives the coefficients when the almost character R_ϕ is expressed in terms of the irreducible unipotent characters $\chi^{\mathcal{F}}_{(x,\sigma)}$.
- As a corollary, equation 1 from the exercise above allows one to express the generalized Deligne-Lusztig characters R_w in terms of the irreducible unipotent characters $\chi^{\mathcal{F}}_{(x,\sigma)}$.

4 Topological field theory perspective

Let $\mathcal{C}$ denote the 3-category whose objects are monoidal categories over $\mathbb{C}$, 1-morphisms are bimodule categories, 2-morphisms are functors of bimodule categories, and 3-morphisms are natural transformations of functors of bimodule categories. As a consequence of the cobordism hypothesis, one can show that any finite group Γ defines a topological field theory Z_Γ valued in $\mathcal{C}$ with

$$\begin{aligned} Z_\Gamma :\ \mathrm{pt} &\mapsto \mathsf{Rep}(\Gamma) = \mathrm{Vec}(\mathrm{pt}/\Gamma), \\ S^1 &\mapsto Z(\mathsf{Rep}(\Gamma)) = \mathrm{Vec}(\frac{\Gamma}{\Gamma}) \\ \Sigma &\mapsto \mathbb{C}[\mathsf{Loc}_\Gamma \Sigma] \\ M &\mapsto \#\mathsf{Loc}_\Gamma M \end{aligned}$$

We explain our notation:

- To a point, Z_Γ assigns the monoidal category $\mathsf{Rep}(\Gamma)$ of representations of Γ, which can also be regarded as the category of vector bundles on the groupoid pt/Γ.
- To a circle, Z_Γ assigns the Drinfeld center of $\mathsf{Rep}(\Gamma)$, which can be identified with the category of Γ-equviariant vector bundles on Γ, where Γ acts on itself by conjugation. This category can also be described as the category of vector bundles on the groupoid $\mathsf{Loc}_\Gamma S^1$ of local systems on the circle S^1.
- To a surface Σ, Z_Γ assigns the space of complex-valued functions from the set of (equivalence classes of) Γ-local systems on Σ.
- To a 3-manifold M, Z_Γ assigns the number of Γ-local systems P on M, each counted with multiplicity $1/|\operatorname{Aut}(P)|$.

It is well known that, for a 2-dimensional topological field theory valued in the category of vector spaces over $\mathbb{C}$, the vectorspace assigned to the circle carries a Frobenius algebra structure. The multiplication comes from the cobordism given by the 'pair of pants'; the unit and trace come from the cobordisms given by a 'cup' and a 'cap' pointing in opposite directions. [Some pictures would be helpful here.]

Now consider the TFT Z_Γ above, and it particular its value on the 2-torus $T = S^1 \times S^1$. The same 'pair of pants' and 'caps' cobordisms, crossed with S^1 endow the vectorspace $Z_\Gamma(T)$ with the structure of a Frobenius algebra. Observe that

$$\mathsf{Loc}_\Gamma T = \{\pi_1(T) \to \Gamma\}/\Gamma = \{(x,y) \in \Gamma^2 \mid [x,y] = 1\}/\Gamma$$

where we quotient by the action of Γ by simultaneous conjugation. In other words, $Z_\Gamma(T)$ can be identified with the space of functions on the set of pairs of commuting elements of Γ, up to simultaneous conjugation. Another name for this set of functions is the set of **2-class functions** on Γ, and we adopt the notation $\mathrm{Cl}^2(\Gamma)$. The Frobenius algebra structure can be written explicitly:

$$\begin{aligned}
\text{multiplication: } \alpha * \beta(x,y) &= \sum_{z \in C_\Gamma(x)} \alpha(x,z)\beta(x,z^{-1}y) \\
\text{unit: } e(x,y) &= \begin{cases} 1 & \text{if } y = 1, \text{ the unit of } \Gamma \\ 0 & \text{otherwise} \end{cases} \\
\text{trace: } \mathrm{tr}(\alpha) &= \sum_{i=1}^{r} \frac{1}{|C_\Gamma(x_i)|}\alpha(x_i,1) \\
\text{inner product: } \langle \alpha, \beta \rangle &= \sum_{i=1}^{r} \frac{1}{|C_\Gamma(x_i)|} \sum_{z \in C_\Gamma(x)} \alpha(x_i,z)\beta(x_i,z^{-1})
\end{aligned}$$

Here, $\{x_1, \ldots, x_n\}$ are representatives for the conjugacy classes of Γ. In fact, one can show that there is an isomorphism of Frobenius algebras:

$$\mathrm{Cl}^2(\Gamma) \simeq \bigoplus_{i=1}^{r} \mathrm{Cl}(C_\Gamma(x_i))$$

where the class functions on each centralizer $C_\Gamma(x_i)$ is endowed with the usual Frobenius algebra structure (which can also be interpreted as arising from a TFT).

Recall that $M(\Gamma)$ is defined as the set of all pairs (x, σ) where x is a conjugacy class representative and σ is an irreducible character of the centralizer $C_\Gamma(x)$. The isomorphism above reveals an orthonormal basis for $\mathrm{Cl}^2(\Gamma)$, indexed by the set $M(\Gamma)$, namely, for $(x, \sigma) \in M(\Gamma)$, define

$$\alpha_{(x,\sigma)}(x', y') = \begin{cases} \sigma(gy'g^{-1}) & \text{if } gx'g^{-1} = x \text{ for some } g \in \Gamma \\ 0 & \text{otherwise} \end{cases}.$$

More simply, $\alpha_{(x_i,\sigma)}(x_j, y) = \sigma(y)\delta_{ij}$.

Now there is a natural involution on the set $\mathrm{Cl}^2(\Gamma)$ of 2-class functions given by switching the places of the two inputs:

$$\begin{aligned}
I : \mathrm{Cl}^2(\Gamma) &\to \mathrm{Cl}^2(\Gamma) \\
\alpha &\mapsto [(x,y) \mapsto \alpha(y,x)].
\end{aligned}$$

One can interpret this involution as switching the generators of the torus. Now, the orthonormal basis $\{\alpha_{(x,\sigma)}\}_{(x,\sigma)\in M(\Gamma)}$ is sent to a new basis $\{I(\alpha_{(x,\sigma)})\}$. It turns out that the change-of-basis matrix is given by Lusztig's non-ableian Fourier tranform that we encountered above:

$$\langle I(\alpha_{(x_1,\sigma_1)}), \alpha_{(x_2,\sigma_2)} \rangle = \frac{1}{|C_\Gamma(x_1)||C_\Gamma(x_2)|} \sum_{\substack{g \in \Gamma \\ gx_1g^{-1} \in C_{\Gamma(\mathcal{F})}(x_2)}} \sigma_1(g^{-1}x_2g)\sigma_2(gx_1^{-1}g^{-1}).$$

It is not clear at the moment why these two formulas appear in such different contexts, but there is some hope that a more direct connection can be made.

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