Notes for 'Representations of Finite Groups'

Iordan Ganev

13 August 2012

Eugene, Oregon

Contents

1 Introduction

The following notes were written in preparation for the first talk of a week-long workshop on categorical representation theory[1]. We focus on basic constructions in the representation theory of finite groups. The participants are likely familiar with much of the material in this talk; we hope that this review provides perspectives that will precipitate a better understanding of later talks of the workshop.

2 Functions on finite sets

Let X be a finite set of size n. Let $\mathbb{C}[X]$ denote the vector space of complex-valued functions on X. In what follows, $\mathbb{C}[X]$ will be endowed with various algebra structures, depending on the nature of X. The simplest algebra structure is pointwise multiplication, and in this case we can identify $\mathbb{C}[X]$ with the algebra $\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$ (n times). To emphasize pointwise multiplication, we write $(\mathbb{C}[X], \text{ptwise})$.

A $\mathbb{C}[X]$-module is the same as X-graded vector space, or a vector bundle on X. To see this, let V be a $\mathbb{C}[X]$-module and let $\delta_x \in \mathbb{C}[X]$ denote the delta function at x. Observe that

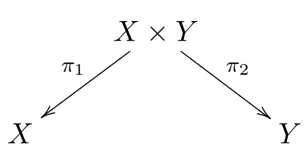
$$\delta_x \cdot \delta_y = \begin{cases} \delta_x & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

It follows that each δ_x acts as a projection onto a subspace V_x of V and $V_x \cap V_y = 0$ if $x \neq y$. Since $1 = \sum_{x \in X} \delta_x$, we have that $V = \bigoplus_{x \in X} V_x$.

Let Y be another finite set and $\alpha : X \to Y$ a map of sets. The **pullback** α^* of α is defined by precomposition with α, while the **pushforward** α_* is defined using summation over the fibers of α:

$$\begin{aligned} \alpha^* : \mathbb{C}[Y] \to \mathbb{C}[X], &\quad \alpha^*(f) = f \circ \alpha \\ \alpha_* : \mathbb{C}[X] \to \mathbb{C}[Y], &\quad \alpha_*(f) : y \mapsto \sum_{x \in \alpha^{-1}(y)} f(x). \end{aligned}$$

For finite sets X and Y, consider the linear map $\Phi : \mathbb{C}[X \times Y] \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$ taking $A \in \mathbb{C}[X \times Y]$ to the integral transform $f \mapsto \sum_{x \in X} A(x, -) f(x)$. In terms of the projections



the map Φ can be written as $A \mapsto (\pi_2)_*(A \cdot \pi_1^*(-))$. On the other hand, given any linear map $\psi \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$, define an element of $\mathbb{C}[X \times Y]$ by sending (x, y) to $\psi(\delta_x)(y)$; thus we have a map $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y]) \to \mathbb{C}[X \times Y]$. It is straightforward to check that these are mutual inverse maps, hence

$$\Phi : \mathbb{C}[X \times Y] \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[X], \mathbb{C}[Y])$$

[1]The talk was given on 13 August 2012; these notes were last updated 31 August 2012.

is an isomorphism. It also true that $\mathbb{C}[X] \otimes \mathbb{C}[Y] \simeq \mathbb{C}[X \times Y]$.

In the case that $X = Y$, the set $\mathbb{C}[X \times X]$ can be given multiplication making Φ an isomorphism of algebras. Consider the projections

$$\begin{array}{ccccc} & & X \times X \times X & & \\ & \swarrow^{\pi_{1,2}} & \downarrow^{\pi_{1,3}} & \searrow^{\pi_{2,3}} & \\ X \times X & & X \times X & & X \times X \end{array}$$

If A and B are complex-valued functions on $X \times X$, then define $A * B = (\pi_{1,3})_*(\pi_{1,2}^* A \cdot \pi_{2,3}^* B)$. To be explicit:

$$A * B(x, y) = \sum_{z \in X} A(x, z) B(z, y).$$

It is almost immediate from this last formula that $(\mathbb{C}[X \times X], *)$ is isomorphic to the matrix algebra $M_n(\mathbb{C}) \simeq \mathrm{End}_{\mathbb{C}}(\mathbb{C}[X])$.

Recall that two rings R and S are **Morita equivalent** if the category R-Mod of R-modules is equivalent to the category S-Mod of S-modules.

Proposition 1. *The algebra* $(\mathbb{C}[X \times X], *)$ *is Morita equivalent to* $\mathbb{C}$.

Proof. We may assume that $X = \{1, 2, \ldots, n\}$ and identify the delta function at (i, j) in $\mathbb{C}[X \times X]$ with the matrix $E_{i,j} \in M_n(\mathbb{C})$ having 1 as its (i, j)-entry and zero elsewhere.

Consider the diagonal map $(\mathbb{C}[X], \text{ptwise}) \to (\mathbb{C}[X \times X], *)$ sending δ_i to $E_{i,i}$. Since this map is an algebra homomorphism, any $\mathbb{C}[X \times X]$-module V also has the structure of a $\mathbb{C}[X]$-module, i.e. a vector bundle on X. Write $V = \bigoplus V_i$. We leave it as an exercise to check that $E_{i,j}$ gives an isomorphism between V_i and V_j. Therefore, the data of a $\mathbb{C}[X \times X]$-module is given by a single vector space.

Conversely, given a vector space W, let $V = X \times W$ be the trivial vector bundle on X. Then V carries a natural action of $\mathbb{C}[X \times X]$. Specifically, the basis element $E_{i,j}$ of $\mathbb{C}[X \times X]$ acts as the composition

$$V \twoheadrightarrow V_i = \{i\} \times W \to \{j\} \times W = V_j \hookrightarrow V,$$

where the first map is the identity on the fiber V_i over i and sends all other fibers to $0 \in V_i \simeq W$, the second map is the identity on W, and the third map is inclusion. $\square$

In this way, we have proven the standard result that all matrix algebras over $\mathbb{C}$ are Morita equivalent to $\mathbb{C}$ using somewhat geometric techniques. Now we consider a generalization. Let $\alpha : X \to Y$ be a surjective function between finite sets. Consider the fiber product

$$X \times_Y X = \{(x_1, x_2) \in X \times X \ : \ \alpha(x_1) = \alpha(x_2)\}.$$

Replacing '$\times$' with '$\times_Y$' in the diagram displaying projections from $X \times X \times X$, we observe that $\mathbb{C}[X \times_Y X]$ is a subalgebra of $(\mathbb{C}[X \times X], *)$. The corresponding subalgebra of $M_n(\mathbb{C})$ consists of block diagonal matrices of the following form: there is one block for each element y of Y, and its size is given by the size of $\alpha^{-1}(y)$. Arguments similar to those in the proof of Proposition 1 can be used to prove the following:

Proposition 2. *The algebra* $(\mathbb{C}[X \times_Y X], *)$ *is Morita equivalent to* $\mathbb{C}[Y]$.

3 The group algebra $\mathbb{C}[G]$

Let G be a finite group. Consider the diagram

$$\begin{array}{ccccc} & & G \times G & & \\ & \overset{\pi_1}{\swarrow} & \downarrow m & \overset{\pi_2}{\searrow} & \\ G & & G & & G \end{array} \tag{1}$$

where the middle arrow is the multiplication map on G. Endow $\mathbb{C}[G]$ with a convolution product: $f_1 * f_2 = m_*(\pi_1^* f_1 \cdot \pi_2^* f_2)$, that is,

$$f_1 * f_2(g) = \sum_{xy=g} f_1(x) f_2(y) = \sum_{x \in G} f_1(x) f_2(x^{-1}g).$$

The **group algebra** of G is defined as $(\mathbb{C}[G], *)$. We make some elementary observations about the group algebra. A basis for $\mathbb{C}[G]$ is given by the delta functions $\{\delta_g : g \in G\}$, and these satisfy the relations $\delta_g * \delta_h = \delta_{gh}$. The multiplicative unit of $\mathbb{C}[G]$ is δ_e, where e is the identity element of G. In particular, every δ_g is invertible in $\mathbb{C}[G]$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G on a complex vector space V. Letting δ_g act by $\rho(g)$ and extending linearly, we see that V acquires the structure of a (left) $\mathbb{C}[G]$-module. Conversely, if V is a $\mathbb{C}[G]$-module, then V carries the structure of a representation of G, where g acts as δ_g. We conclude that there is a bijection between the $\mathbb{C}[G]$-module structures on V and the representations of G on V. In other words, there is an equivalence between the category $\mathbb{C}[G]$-Mod of $\mathbb{C}[G]$-modules and the category $\mathrm{Rep}(G)$ of complex representations of G. Moreover, this equivalence commutes with the forgetful functors:

$$\begin{array}{ccc} \mathbb{C}[G]\text{-Mod} & \xrightarrow{\quad\simeq\quad} & \mathrm{Rep}(G) \\ & {\scriptstyle\text{forget}}\searrow \quad \swarrow{\scriptstyle\text{forget}} & \\ & \mathrm{Vec}_{\mathbb{C}} & \end{array}$$

Thinking of G as a finite set, recall that $\mathbb{C}[G \times G]$ has a matrix multiplication, which we now denote $(\mathbb{C}[G \times G], \text{matrix})$. The group G acts diagonally on $\mathbb{C}[G \times G]$ as $(g \cdot A)(h, k) = A(g^{-1}h, g^{-1}k)$. The following proposition allows us to realize the convolution product as a matrix multiplication.

Proposition 3. *The space* $\mathbb{C}[G \times G]^G$ *of* G*-invariant functions is a subalgebra of* $(\mathbb{C}[G \times G], \text{matrix})$. *There is an isomorphism of algebras* $(\mathbb{C}[G], *)$ *and* $(\mathbb{C}[G \times G]^G, \text{matrix})$.

Proof. The proof of the first statement is straightforward. We leave the reader to verify that the maps $\mathbb{C}[G] \to \mathbb{C}[G \times G]^G : f \mapsto [(h, k) \mapsto f(h^{-1}k)]$ and $\mathbb{C}[G \times G]^G \to \mathbb{C}[G] : \phi \mapsto [g \mapsto \phi(g^{-1}, 1)]$ are mutual inverses. $\square$

Observe that G acts on $\mathbb{C}[G]$ by conjugation: $f^x(g) = f(x^{-1}gx)$. The **class functions** on G, denoted $\mathbb{C}[G]^G$ or $\mathbb{C}[G/G]$ or $\mathbb{C}[G/\mathrm{ad}G]$, are the fixed points of this action:

$$\mathbb{C}[G/G] = \{f \in \mathbb{C}[G] : f(xgx^{-1}) = f(g) \text{ for all } \; x, g \in G\}.$$

Recall that the **cocenter**, or **abelianization**, of an algebra A over $\mathbb{C}$ is defined as the A-module $A/[A, A]$ where $[A, A]$ is the subspace generated by all elements of the form $ab - ba$. The map $\pi : A \to A/[A, A]$ from A to the cocenter has the following universal property. Suppose V is a vector space and $f : A \to V$ is a linear map with the property that $f(ab) = f(ba)$ for all $a, b \in A$, that is, f is a trace map. Then f factors uniquely through π. For this reason, the quotient map π is called the **universal trace** of A. If $\tilde{\pi} : A \to C$ is another map satisfying the same universal property as π, then we can identify C with the cocenter of A. We leave the proof of the following proposition as an exercise.

Proposition 4. *The class functions $\mathbb{C}[G/G]$ are the center of the group algebra $\mathbb{C}[G]$. Moreover, the projection $\pi : \mathbb{C}[G] \to \mathbb{C}[G/G]$ defined on basis elements by*

$$\pi(\delta_g) = \frac{1}{|G|} \sum_{x \in G} \delta_{xgx^{-1}}.$$

is a universal trace, hence the class functions $\mathbb{C}[G/G]$ can be identified with the cocenter of the group algebra $\mathbb{C}[G]$.

Remark. In later talks we will see that the center of an algebra is its degree 0 Hochschild cohomology and the cocenter is its degree 0 Hochschild homology. Hence we have that $HH_0(\mathbb{C}[G]) = HH^0(\mathbb{C}[G]) = \mathbb{C}[G/G]$.

4 Induced representations

Let G be a finite group and K a subgroup of G. In this case, $\mathbb{C}[K]$ is a subalgebra of $\mathbb{C}[G]$ and any representation of G is a representation of K by restriction. Thus we have a functor

$$\mathrm{Res}_G^K : \mathrm{Rep}(G) \to \mathrm{Rep}(K).$$

Natural questions are: does the functor Res_G^K have a left adjoint? a right adjoint? The answer to both questions turns out to be yes.

A left adjoint to Res_G^K is given by the **induction** functor

$$\begin{aligned} \mathrm{Ind}_K^G : \mathrm{Rep}(K) &\to \mathrm{Rep}(G) \\ W &\mapsto \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W. \end{aligned}$$

Note that $\mathbb{C}[G]$ is a $\mathbb{C}[G]$-$\mathbb{C}[K]$-bimodule via multiplication in the group algebra and that Ind_K^G is an additive functor, i.e. $\mathrm{Ind}_K^G(W_1 \oplus W_2) \simeq \mathrm{Ind}_K^G(W_1) \oplus \mathrm{Ind}_K^G(W_2)$. Thinking of representations as modules for the group algebra, it is straightforward to verify that there are indeed isomorphisms

$$\mathrm{Hom}_K(W, \mathrm{Res}_G^K(V)) \simeq \mathrm{Hom}_G(\mathrm{Ind}_K^G(W), V)$$

functorial in $V \in \mathrm{Rep}(G)$ and $W \in \mathrm{Rep}(K)$.

For example, let $\mathbb{C}_{\mathrm{triv}}$ denote the trivial representation of K. The induced representation of G can be identified with the invariants of $\mathbb{C}[G]$ under the right action of K, or, equivalently, functions on the left cosets G/K. In symbols, $\mathrm{Ind}_K^G(\mathbb{C}_{\mathrm{triv}}) = \mathbb{C}[G/K]$.

Now we describe a right adjoint to the restriction functor. Let $\sigma : K \to \mathrm{GL}(W)$ be a representation of K and define the **coinduction** functor $\mathrm{Coind}_K^G : \mathrm{Rep}(K) \to \mathrm{Rep}(G)$ as

$$\mathrm{Coind}_K^G(W) = \{f : G \to W : f(kg) = \sigma(k)f(g) \;\; \text{for all} \;\; g \in G, k \in K\}.$$

The action of G is given by $(g \cdot f)(x) = f(xg)$. Equivalently, $\mathrm{Coind}_K^G(W) = \mathrm{Hom}_K(\mathbb{C}[G], W)$, where K acts on $\mathbb{C}[G]$ by left multiplication and G acts on $f : \mathbb{C}[G] \to W$ as $(g \cdot f)(\delta_x) = f(\delta_{xg})$.

Proposition 5. *Let V be a representation of G and W a representation of K. Then*

$$\mathrm{Hom}_K(\mathrm{Res}_G^K(V), W) \simeq \mathrm{Hom}_G(V, \mathrm{Coind}_K^G(W)).$$

Consequently, Coind_K^G *is a right adjoint to* Res_G^K.

Proof. The vector space

$$\mathrm{Hom}(V, \mathrm{Hom}(\mathbb{C}[G], W)) \simeq \mathrm{Hom}(V \otimes \mathbb{C}[G], W) \simeq \mathrm{Hom}(\mathbb{C}[G], \mathrm{Hom}(V, W)) \tag{2}$$

admits a left action of G and a right action of K. The two actions commute; taking $G \times K$-invariants on the far left side of equation 2, we obtain:

$$\mathrm{Hom}(V, \mathrm{Hom}(\mathbb{C}[G], W))^{G\times K} = \mathrm{Hom}_G(V, \mathrm{Hom}_K(\mathbb{C}[G], W)) = \mathrm{Hom}_G(V, \mathrm{Coind}_K^G(W)).$$

Taking $G \times K$-invariants on the far right side of equation 2, we obtain:

$$\mathrm{Hom}(\mathbb{C}[G], \mathrm{Hom}(V, W))^{G\times K} = \mathrm{Hom}_G(\mathbb{C}[G], \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W)) = \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W).$$

The proposition now follows. □

Proposition 6. *As representations of G, $\mathrm{Ind}_K^G(W)$ and $\mathrm{Coind}_K^G(W)$ are isomorphic.*

Proof. Let $\sigma : K \to \mathrm{GL}(W)$ be the the group homomorphism giving the action of K on W. Define a linear map $\epsilon : \mathbb{C}[G] \times W \to \mathrm{Hom}(\mathbb{C}[G], W)$ by

$$\epsilon(\delta_x, w)(\delta_y) = \begin{cases} \sigma(yx)w & \text{if } yx \in K \\ 0 & \text{otherwise} \end{cases}$$

and extending linearly. One shows that the map $\epsilon(\delta_x, w)$ is K-equivariant and that ϵ is $\mathbb{C}[K]$-bilinear, so we obtain a map

$$\epsilon : \mathbb{C}[G] \otimes_{\mathbb{C}[K]} W \to \mathrm{Hom}_{\mathbb{C}[K]}(\mathbb{C}[G], W).$$

It is not difficult to see that ϵ is G-equivariant.

Fix a set of left coset representatives $\{g_1, \ldots, g_n\}$ for K in G. Given $\phi \in \mathrm{Hom}_{\mathbb{C}[K]}(\mathbb{C}[G], W)$, a computation shows that $\epsilon(\sum_i \delta_{g_i} \otimes \phi(g_i^{-1})) = \phi$, and this proves that ϵ is surjective. Note that $\mathbb{C}[G] \otimes_{\mathbb{C}[K]} W = \bigoplus_i \delta_{g_i} \otimes W$; therefore, to show that ϵ is injective, it suffices to show that $\epsilon(\sum_i \delta_{g_i} \otimes w_i)$ is zero if and only if w_i is zero for all i. For each i, let $\alpha(i) \in \{1, \ldots, n\}$ be the unique index such that $g_i^{-1}K = g_{\alpha(i)}K$. Since the map $\epsilon(\sum_i \delta_{g_i} \otimes w_i)$ is K-equivariant, it is determined by its values on $\{\delta_{g_j}\}$. Direct computations verify that $\epsilon(\sum_i \delta_{g_i} \otimes w_i) = 0$ if and only if for all j we have

$$\sum_{\{i : g_j g_i \in K\}} \sigma(g_j g_i) w_i = 0$$

which happens if and only if $\sigma(g_j g_{\alpha(j)}) w_{\alpha(j)} = 0$ for all j. The fact that $\sigma(g_j g_{\ alpha(j)})$ is invertible and α is a bijection imply that $w_i = 0$ for all i. $\square$

Consequently, we have a single induction functor that is left and right adjoint to the restriction functor. These adjunctions are known as **Frobenius reciprocity**:

$$\mathrm{Hom}_G(\mathrm{Ind}_K^G(W), V) \simeq \mathrm{Hom}_K(W, \mathrm{Res}_G^K(V))$$

$$\mathrm{Hom}_G(V, \mathrm{Ind}_K^G(W)) \simeq \mathrm{Hom}_K(\mathrm{Res}_G^K(V), W).$$

In particular, both restriction and induction are exact functors.

Finally, we give a geometric picture of induced representations. Let W be a representation of K. Consider the trivial bundle $G \times W \to G$ on G. Let $G \times_K W$ be the 'balanced product' formed by taking the quotient of $G \times W$ by the equivalence relation $(g, w) \sim (gk, k^{-1} \cdot w)$ for any k in K. The map $G \times_K W \to G/K$ sending $[g, w]$ to the coset gK is well-defined and makes $G \times_K W$ a vector bundle over G/K. The induced representation $\mathrm{Ind}_K^G(W)$ can be defined as global sections of $G \times_K W$. Note that $G \times_K W \to G/K$ is trivial W-bundle over G/K once a complete set of coset representatives is chosen. The group G acts on the G/K by changing coset representatives, and this gives an action of G on the space of sections. We leave the details to the reader.

5 The Hecke algebra $\mathcal{H}(G, K)$

As in the previous section, let G be a finite group and K a subgroup of G. Let V be a representation of G and define V^K as the subspace of K-invariant vectors:

$$V^K = \{v \in V : k \cdot v = v \ \text{ for all } \ k \in K\}.$$

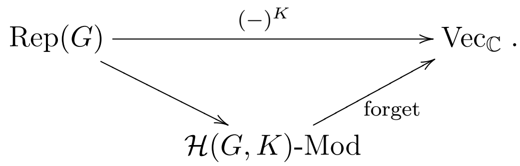
Observe that taking K-invariants is functorial, so we have a functor $(-)^K : \mathrm{Rep}(G) \to \mathrm{Vec}_{\mathbb{C}}$. By Frobenius reciprocity, this functor is representable by $\mathbb{C}[G/K]$:

$$V^K = \mathrm{Hom}_K(\mathbb{C}_{\mathrm{triv}}, \mathrm{Res}(V)) = \mathrm{Hom}_G(\mathrm{Ind}(\mathbb{C}_{\mathrm{triv}}), V) = \mathrm{Hom}_G(\mathbb{C}[G/K], V),$$

where $\mathbb{C}_{\mathrm{triv}}$ denotes the trivial representation of K. Since there is no danger of confusion, we have abbreviated the induction and restriction functors as Ind and Res. There is a natural right action of the algebra $\mathrm{End}_G(\mathbb{C}[G/K])$ on every $V^K = \mathrm{Hom}_G(\mathbb{C}[G/K], V)$ by precomposition:

$$(\phi, \alpha) \mapsto \phi \circ \alpha$$

for all $\alpha \in \mathrm{End}_G(\mathbb{C}[G/K])$ and $\phi \in \mathrm{Hom}_G(\mathbb{C}[G/K], V)$. The algebra $\mathcal{H}(G, K) := \mathrm{End}_G(\mathbb{C}[G/K])$ is known as the **Hecke algebra** of the pair (G, K). By the above comments, there is a factorization



The Yoneda lemma implies that $\mathcal{H}(G, K)^{\mathrm{op}} \simeq \mathrm{End}((-)^K)$.

Let $\langle \mathbb{C}[G/K] \rangle$ be the full subcategory of $\mathrm{Rep}(G)$ consisting of representations V such that $\mathrm{Hom}_G(\mathbb{C}[G/K], V) \neq 0$ (equivalently, $V^K \neq 0$), together with the zero representation. This is often referred to as the **subcategory generated** by $\mathbb{C}[G/K]$.

Proposition 7. *There is an equivalence of categories* $\langle \mathbb{C}[G/K] \rangle \simeq \mathcal{H}(G, K)$-Mod.

Proof. The Barr-Beck theorem provides one way to see this equivalence. Since the Barr-Beck theorem will feature in a later talk, the reader may wish to read this proof after learning the Barr-Beck theorem.

A left adjoint to the exact functor $(-)^K$ is the composition $\mathrm{Vec}_{\mathbb{C}} \to \mathrm{Rep}(K) \xrightarrow{\mathrm{Ind}} \mathrm{Rep}(G)$ where the first functor is the inclusion of vector spaces as the full subcategory of trivial representations. The corresponding monad on $\mathrm{Vec}_{\mathbb{C}}$ is given by tensoring with $\mathcal{H}(G, K)$ since

$$\mathbb{C} \mapsto (\mathrm{Ind}(\mathbb{C}_{\mathrm{triv}}))^K = \mathbb{C}[G/K]^K = \mathrm{Hom}(\mathbb{C}[G/K], \mathbb{C}[G/K]) = \mathcal{H}(G, K)$$

and extending additively. Since $V^K \neq 0$ for nonzero objects V of $\langle \mathbb{C}[G/K] \rangle$, the Barr-Beck theorem immediately implies the result. □

Therefore, the Hecke algebra allows us to probe into the category of representations of G. If K is small, then many representations of G will have K-invariants, so knowledge of $\mathcal{H}(G, K)$ and its category of modules is more valuable. However, in this case $\mathcal{H}(G, K)$ may be more difficult to understand. If K is large, then G/K is small and $\mathcal{H}(G, K)$ may have a simpler structure, for example it may be commutative. The disadvantage is that in this case we may acquire less information about the category $\mathrm{Rep}(G)$.

Let ${}^K\mathbb{C}[G]^K$ denote the left and right K-invariant functions in $\mathbb{C}[G]$. It is easy to see that ${}^K\mathbb{C}[G]^K$ can be identified with functions on the double cosets $\mathbb{C}[K\backslash G/K]$. In certain contexts, the Hecke algebra $\mathcal{H}(G, K)$ is defined as $\mathbb{C}[K\backslash G/K]$; this is justified by the following proposition.

Proposition 8. *The space* ${}^K\mathbb{C}[G]^K = \mathbb{C}[K\backslash G/K]$ *is a subalgebra of* $\mathbb{C}[G]$ *isomorphic to* $\mathcal{H}(G, K)$.

Instead of providing a detailed proof, we mention several ways to gain insight on the proposition. Recall that, in the definition of the multiplication in the group algebra, we considered a diagram with maps out of $G \times G$. The addition of appropriate quotients yields the following diagram, whose maps are well-defined:

$$\begin{array}{ccccc} & & K\backslash G \times_K G/K & & \\ & \swarrow^{\pi_1} & \downarrow{\scriptstyle m} & \searrow^{\pi_2} & \\ K\backslash G/K & & K\backslash G/K & & K\backslash G/K \end{array}$$

Here '$\times_K$' again denotes the balanced product, as defined in the previous section. One can use this diagram to deduce that $\mathbb{C}[K\backslash G/K]$ is an algebra under convolution.

In order to demonstrate the isomorphism of $\mathbb{C}[K\backslash G/K]$ with the Hecke algebra, we can use the definition of the Hecke algebra and the representability of the functor $(-)^K$ to obtain isomorphisms of vector spaces

$$\mathcal{H}(G, K) = \mathrm{Hom}_G(\mathbb{C}[G/K], \mathbb{C}[G/K]) \simeq \mathbb{C}[G/K]^K \simeq \mathbb{C}[K\backslash G/K],$$

that are in fact isomorphisms of algebras.

Alternatively, consider the diagonal action of G on $\mathbb{C}[G/K \times G/K]$. In a manner similar to the above discussion of the group algebra, there are algebra isomorphisms

$$\mathcal{H}(G, K) \simeq \mathbb{C}[G/K \times G/K]^G \simeq \mathbb{C}[G\backslash(G/K \times G/K)] \simeq \mathbb{C}[K\backslash G/K].$$

Here we use the (easily verified) fact that the orbits of $G/K \times G/K$ under the diagonal action of G can be identified with the double coset space $K\backslash G/K$.

Another approach is to use the idempotents: it is a general fact that in any algebra A with an idempotent element e, the set eAe is a subalgebra isomorphic to $\mathrm{End}_A(Ae)$. In our case, take $A = \mathbb{C}[G]$ with the idempotent $e_K = \frac{1}{|K|}\sum_{k\in K}\delta_k$. Simple computations show that $\mathbb{C}[G/K] = \mathbb{C}[G] * e_K$ and $\mathbb{C}[K\backslash G/K] = e_K * \mathbb{C}[G] * e_K$. Hence $\mathrm{End}_G(\mathbb{C}[G/K]) \simeq \mathbb{C}[K\backslash G/K]$.

To conclude this section, we describe a more general formulation of the Hecke algebra. Let W be an irreducible representation of K. Define the Hecke algebra of the triple (G, K, W) as $\mathcal{H}(G, K, W) = \mathrm{End}_G(\mathrm{Ind}(W))$. Consider the functor $\mathrm{Rep}(G) \to \mathrm{Vec}_{\mathbb{C}}$ taking a representation V to its "W-isotypic component" under K, that is, the largest subrepresentation of $\mathrm{Res}(V)$ isomorphic to some number of copies of W. Using identical arguments as above, one can see that this functor is representable by $\mathrm{Ind}(W)$ and establishes an equivalence between that category of $\mathcal{H}(G, K, W)$-modules and the full subcategory $\langle \mathrm{Ind}(W)\rangle$ of $\mathrm{Rep}(G)$.

6 Characters and the Frobenius character formula

Let V be a finite dimensional representation of a finite group G. Consider the 'matrix coefficients' map

$$\begin{aligned} \phi : \mathrm{End}(V) \simeq V^* \otimes V &\to \mathbb{C}[G] \\ v^* \otimes v &\mapsto [g \mapsto \langle v^*, g \cdot v\rangle]. \end{aligned}$$

The **character** of G on V is defined as the element $\chi_V := \phi(\mathrm{Id}_V) \in \mathbb{C}[G]$. For any fixed basis $\{e_i\}$ of V, the element $\mathrm{Id}_V \in \mathrm{End}(V)$ corresponds to $\sum e_i^* \otimes e_i \in V^* \otimes V$. So

$$\chi_V(g) = \sum_i e_i^*(g \cdot e_i) = \mathrm{tr}(e_j^*(g \cdot e_i)) = \mathrm{tr}(\rho(g))$$

where $\rho(g) = (e_j^*(g \cdot e_i))_{i,j}$ is the matrix giving the action of g on V in the basis $\{e_i\}$. Since the trace function on matricies is a class function, it follows that χ_V is also a class function, i.e. and element of $\mathbb{C}[G/G]$.

From now on, all representations of G are assumed to be finite dimensional. Let $V_1, \ldots, V_r$ be the irreducible representations of G, with characters $\chi_1, \ldots, \chi_r$. We review some basic facts about characters without proof.

1. There is a non-degenerate Hermitian inner product on the space of class functions $\mathbb{C}[G/G]$ given by
$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$
(This is nothing more than the 'dot product' that defines a non-degenerate Hermitian inner product on any finite dimensional complex vector space.)

2. The characters $\chi_1, \ldots, \chi_r$ form an orthonormal basis for $\mathbb{C}[G/G]$. In particular, the number of irreducible representations of a finite group equals the number of conjugacy classes. Since any representation of G decomposes as a direct sum of irreducibles, we have further that a representation is determined by its character.

3. Let V and U be representations of G. Then $\chi_{V \oplus U} = \chi_V + \chi_U$ and $\chi_{V \otimes U} = \chi_V \cdot \chi_U$ (pointwise). Also,
$$\langle \chi_V, \chi_U \rangle = \dim \mathrm{Hom}_G(V, U).$$
Let K be a subgroup of G. We use the notation ψ_W for the character of a representation W of K. Therefore, Frobenius reciprocity implies that
$$\langle \chi_{\mathrm{Ind}(W)}, \chi_V \rangle = \langle \psi_W, \psi_{\mathrm{Res}(V)} \rangle \quad \text{and} \quad \langle \chi_V, \chi_{\mathrm{Ind}(W)} \rangle = \langle \psi_{\mathrm{Res}(V), \psi_W} \rangle.$$

4. As algebras, $\mathbb{C}[G] \simeq \bigoplus \mathrm{End}(V_i)$. The idempotents are
$$e_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \cdot \delta_g \in \mathbb{C}[G]$$
for $1 \le i \le r$. On a representation U of G, the element e_i acts as a projection onto the isotypic component of U corresponding to V_i.

We conclude with a discussion of the Frobenius character formula. Let K be a subgroup of G. Then the following diagram commutes:

$$\begin{array}{ccc} K & \xrightarrow{i} & G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ K/K & \xrightarrow{\pi} & G/G \end{array}$$

where the top map is the inclusion, the vertical maps are the quotients by the action of conjugation, and the bottom map sends the conjugacy class of an element k of K to its conjugacy class in G. We have induced maps

$$\mathbb{C}[K/K] \xrightarrow{p^*} \mathbb{C}[K] \xrightarrow{i_*} \mathbb{C}[G] \xrightarrow{q_*} \mathbb{C}[G/G]$$

defined as follows. The map p^* is the usual pullback, that is, precomposition with the quotient map p. The map i_* is defined on basis elements as $i_*(\delta_k) = [G:K]\delta_k$, so is a scaled version of the usual pushforward. Finally, the map q_* is the projection to class functions that we saw in section 3; it is defined on basis elements as

$$q_*(\delta_g) = \frac{1}{|G|}\sum_{x\in G}\delta_{xgx^{-1}}.$$

Define the pushforward $\pi_* : \mathbb{C}[K/K] \to \mathbb{C}[G/G]$ to be the composition $q_* \circ i_* \circ p_*$. Explicity, π_* takes the form

$$\begin{aligned}\pi_* : \mathbb{C}[K/K] &\to \mathbb{C}[G/G] \\ f &\mapsto [g \mapsto \frac{1}{|K|}\sum_{x\in G}\dot{f}(xgx^{-1})]\end{aligned}$$

where $\dot{f} \in \mathbb{C}[G]$ conincides with f on K and is 0 otherwise. If W is a representation of K with character ψ, then we abbreviate by $\mathrm{Ind}(\psi)$ the character of the induced representation $\mathrm{Ind}_K^G(W)$. The following result is known as the **Frobenius character formula**:

Proposition 9. *Let ψ be the character of a representation W of K. Then $\mathrm{Ind}(\psi) = \pi_*\psi$. Explicitly,*

$$\mathrm{Ind}(\psi)(g) = \frac{1}{|K|}\sum_{x\in G}\dot{\psi}(xgx^{-1}),$$

where $\dot{f} \in \mathbb{C}[G]$ conincides with f on K and is 0 otherwise.

Proof. Let $\eta \in \mathbb{C}[G/G]$ be arbitrary. Then

$$\begin{aligned}\langle \eta, \pi_*\psi\rangle &= \frac{1}{|G|}\sum_{g\in G}\overline{\eta(g)}\cdot \pi_*\psi(g) = \frac{1}{|G||K|}\sum_{g\in G}\sum_{x\in G}\overline{\eta(g)}\cdot\dot{\psi}(xgx^{-1}) \\ &= \frac{1}{|G||K|}\sum_{g\in G}\sum_{k\in K}\sum_{\substack{x\in G\\ xgx^{-1}=k}}\overline{\eta(g)}\cdot\psi(k) = \frac{1}{|G||K|}\sum_{k\in K}\sum_{x\in G}\sum_{\substack{g\in G\\ g=x^{-1}kx}}\overline{\eta(g)}\cdot\psi(k) \\ &= \frac{1}{|G||K|}\sum_{k\in K}\sum_{x\in G}\overline{\eta(x^{-1}kx)}\cdot\psi(k) = \frac{1}{|G||K|}\sum_{k\in K}\sum_{x\in G}\overline{\eta(k)}\cdot\psi(k) \\ &= \frac{|G|}{|G||K|}\sum_{k\in K}\overline{\eta(k)}\cdot\psi(k) = \langle \mathrm{Res}(\eta), \psi\rangle = \langle \eta, \mathrm{Ind}(\psi)\rangle.\end{aligned}$$

The first three equalities follow from the definitions of $\langle , \rangle$, $\pi_*\psi$, and $\dot{\psi}$. The sixth equality uses the fact that η is a class function, and the last equality invokes Frobenius reciprocity. Since η is arbitrary, the result follows from the non-degeneracy of the inner product on $\mathbb{C}[G/G]$. □

We describe another perspective on this result. Recall that the **Grothendieck group** of an (essentially small) abelian category $\mathcal{C}$ is defined as the free abelian group on the set $\{[X]\}$ of isomorphism classes of objects of $\mathcal{C}$ modulo the relation $[Y] = [X] + [Z]$ for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$. The Grothendieck group of $\mathcal{C}$ is denoted $K(\mathcal{C})$. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories, then F gives rise to a homomorphism $\tilde{F} : K(\mathcal{C}) \to K(\mathcal{D})$ between the Grothendieck groups defined by $\tilde{F}([X]) = [F(X)]$.

The facts listed earlier in this section imply that the complexified Grothendieck group $K(\mathrm{Rep}_f(G)) \otimes_{\mathbb{Z}} \mathbb{C}$ of the category $\mathrm{Rep}_f(G)$ of finite-dimensional complex representations of G can be identified with the vector space $\mathbb{C}[G/G]$ of class functions[2]. Passing from $\mathrm{Rep}_f(G)$ to $K(\mathrm{Rep}_f(G)) \otimes \mathbb{C}$ replaces a representation by its character. Now let K be a subgroup of G. The induction and restriction functors

$$\mathrm{Rep}_f(K) \underset{\mathrm{Res}}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathrm{Rep}_f(G)$$

give linear maps

$$\mathbb{C}[K/K] \rightleftarrows \mathbb{C}[G/G] \,.$$

We claim that these linear maps are π_* and π^*, where $\pi : K/K \to G/G$ is the function from earlier in this section. Here π^* dontes the ususal pullback, i.e. precomposition with π, whereas π_* is the (special) pushforward defined above. In other words, on the level of characters, $\mathrm{Ind} = \pi_*$ and $\mathrm{Res} = \pi^*$. More precisely:

Proposition 10. *Let V be a representation of G and W a representation of K. Then $\psi_{\mathrm{Res}(V)} = \pi^*(\chi_V)$ and $\chi_{\mathrm{Ind}(W)} = \pi_*(\psi_W)$.*

The first equality is easy since $\pi^*(\chi_V) = \chi_V|_K$, while second equality is just Proposition 9.

7 Exercises

1. Complete the proof of Proposition 1 by showing that $E_{i,j}$ gives an isomorphism between V_i and V_j. Prove Proposition 2 by adopting arguments from the proof of Proposition 1.
2. Let X and Y be finite sets. Show that $\mathbb{C}[X \times Y] = \mathbb{C}[X] \otimes_{\mathbb{C}} \mathbb{C}[Y]$. More generally, show that $\mathbb{C}[X \times_Z Y] = \mathbb{C}[X] \otimes_{\mathbb{C}[Z]} \mathbb{C}[Y]$ where Z is a finite set and with maps $X \to Z$ and $Y \to Z$.
3. Complete the proof of Proposition 3 by showing that the maps given in the text are mutual inverses.
4. Provide a proof for Propostion 4.
5. Let K be a subgroup of a finite group G and W a representation of K. Show that the representation of G on sections of the bundle $G \times_K W \to G/K$ is isomorphic to the representation $\mathrm{Coind}_K^G(W)$ (and hence also to $\mathrm{Ind}_K^G(W)$).
6. Provide a detailed proof of Proposition 8.

[2]In fact, $\mathrm{Rep}_f(G)$ is a tensor category, so $K(\mathrm{Rep}_f(G)) \otimes \mathbb{C}$ is an algebra, and it is isomorphic to ($\mathbb{C}[G/G]$, ptwise), but this extra structure is not relevant for the present discussion.

7. Let V be a finite dimensional representation of G. This exercise gives another way to see that χ_V is a class function. Suppose first that V is irreducible. Use Schur's lemma to prove that $\mathrm{End}(V)^G = \mathbb{C} \cdot \mathrm{Id}_V$. Prove that the map $\phi : \mathrm{End}(V) \to \mathbb{C}[G]$ dicussed in the text is G-equivariant for the action of G on $\mathbb{C}[G]$ by conjugation, and conclude that $\phi(\mathrm{Id}_V)$ is a class function. Use the complete reducibility of finite dimensional representations to prove the result for arbitrary V.

8. If G acts on a set X, show that $\mathbb{C}[X]$ carries the structure of a representation of G. Assume X is finite. Prove that $\chi_{\mathbb{C}[X]}$ counts fixed points: $\chi_{\mathbb{C}[X]}(g) = \#\{x \in X : g \cdot x = x\}$. Observe that $\mathcal{H}(G, K)$ acts on $\mathbb{C}[X]^K = \mathbb{C}[K \backslash X]$.

Acknowledgements

Much of the content of these notes is inspired by a course on representation theory taught by David Ben-Zvi during Spring 2012, as well as individual conversations I have had with him. I am very grateful to him, and also to Lee Cohn, David Jordan, Hendrik Orem, and Pavel Safronov for many helpful discussions and suggestions. Special thanks to Alberto García-Raboso for a careful reading and insightful comments.

References

[1] D. Bump. *Lie Groups*. Springer, 2004.

[2] W. Fulton and J. Harris. *Representation Theory: A First Course*. Springer, 1991.