Notes on deformation theory

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Abstract

This document is a summary of the author's lecture notes for a two-part seminar talk on deformation theory. The first part consists mostly of examples and basic constructions in deformation theory; the outline is given in the first section of this document. In second part, we follow the first six sections of Jacob Lurie's ICM address on moduli problems for ring spectra. The reader is warned that these notes contain many vague statements and emphasize motivation and general ideas rather than rigorous development of the theory.

1 Outline of the first part

- Algebra structures on a vector space
- Infinitesimal deformations of an associative algebra
- The Hochschild complex and the Gerstenhaber bracket
- Differential-graded Lie algebras and the Maurer-Cartan equation
- Formal deformations of an associative algebra
- The Hochschild-Kostant-Rosenberg theorem, polyvector fields, and Poisson structures
- Deformation quantization of Poisson structures and Kontsevich's theorem

2 Artin rings and differential graded Lie algebras

Let k be a field. Last time we considered n-th order deformations of an associative k-algebra A using the k-algebra $R = k[\epsilon]/(\epsilon^{n+1})$. This k-algebra is 'close' to k in the sense that

$$R/\mathrm{nil}(R) \simeq k.$$

In other words, R is a local commutative k-algebra whose maximal ideal coincides with the nilpotent radical $\mathrm{nil}(R)$ and whose residue field is k. We refer to any k-algebra satisfying these conditions as a (commutative) *Artin k-algebra*, and adopt the notation Art_k for the category of Artin k-algebras. If R is an Artin k-algebra, the map $\epsilon : R \to R/\mathrm{nil}(R) = k$ is called the *augmentation* of R. Observe that $\mathrm{nil}(R)$ is the unique prime ideal of R, so $\mathrm{Spec}(R)$ is a single point. One can consider $\mathrm{Spec}(R)$ as the point $\mathrm{Spec}(k)$, together with 'nilpotent fuzz'

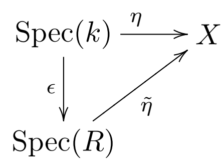
Observe that the algebra $k[[\hbar]]$ of formal power series is not an Artin k-algebra, but is an inverse limit of the Artin k-algebras $k[\hbar]/(\hbar^n)$ for $n \geq 1$.

How do we make use of the nice properties of Artin k-algebras? Let $X : \mathsf{Ring} \to \mathsf{Gpd}$ denote a groupoid-valued functor on the category Ring of commutative rings. For example, X may be the functor of points of a scheme or a stack. Let η be an object of the groupoid $X(k)$. We say that the pair $x = (k, \eta)$ is a *point* of X. For any Artin k-algebra $R \in \mathsf{Art}_k$, the augmentation induces a map of groupoids

$$X(\epsilon) : X(R) \to X(k).$$

The fiber $X(\epsilon)^{-1}(\eta)$ over the distinguished object $\eta \in X(k)$ is defined as the groupoid whose objects are the objects of $X(R)$ that map to η, and whose morphisms are the morphisms in $X(R)$ that map to the identity morphism of η. Heuristically, the fiber encodes the extent to which we can perturb the point x in certain directions within the scheme/ stack/ moduli space X.

For example, if X is a scheme (so that X is valued in sets rather than in groupoids), then the fiber is the set of all $\tilde{\eta} \in X(R)$ making the following diagram commute:



Considering all Artin k-algebras at once, we arrive at the functor

$$\begin{aligned} X_x : \mathsf{Art}_k &\to \mathsf{Gpd} \\ R &\mapsto X(\epsilon)^{-1}(\eta) \end{aligned}$$

Or perhaps, depending on the context, there may be meaningful notion of flatness and it is more relevant to consider the functor

$$X_x^{\text{fl}} : \mathsf{Art}_k \to \mathsf{Gpd}$$

that assigns to R the 'flat points' in the fiber of $X(\epsilon)$ over η. The tangent space to X at x is the value of X_x (or X_x^{fl}) at the ring $k[\epsilon]/(\epsilon^2)$, that is:

$$T_x X = X_x(k[\epsilon]/(\epsilon^2)) \text{ (or } X_x^{\text{fl}}(k[\epsilon]/(\epsilon^2))).$$

Example. Consider the functor $X : \mathsf{Ring} \to \mathsf{Gpd}$ that assigns to a ring S the groupoid whose objects are S-algebras and whose morphisms are isomorphisms of S-algebras. A point $x = (k, \eta)$ is just a k-algebra A and for $R \in \mathsf{Art}_k$, we have

$$X_x(R) = \{R\text{-algebras } A' \text{ equipped with an isomorphism } A' \otimes_R k \simeq A \text{ of } k\text{-algebras}\}.$$

We define the flat points in $X_x(R)$ to be those R-algebras A' that are flat (equivalently free) as R-modules. Then $X_x^{\text{fl}}(R)$ is precisely the groupoid of R-deformations of A that was discussed in the previous lecture. The tangent space at x is the groupoid of infinitesimal deformations of A, which is the set $Z^2(A)$ of Hochschild 2-cocycles together with the action of the Hochschild 1-cochains $C^1(A)$ via the differential. The components can be identified with the second Hochschild cohomology $HH^2(A)$ of A, and the automorphism group of any point is the group $Z^1(A)$ of 1-cocycles. In summary:

$$T_x X = Z^2(A)/C^1(A) \qquad \pi_0(T_x X) = HH^2(A) \qquad \pi_1(T_x X) = Z^1(A).$$

Let R be an Artin k-algebra with maximal ideal $m = \text{nil}(R)$ and let $L^\bullet$ be a differential graded (dg) Lie algebra. Then $L^0 \otimes m$ is a nilpotent Lie algebra, so we can exponentiate to obtain a group

$\exp(L^0 \otimes m)$. This group acts on the set $\mathrm{MC}(L \otimes m)$ of Maurer-Cartan elements in the dg Lie algebra $L \otimes m$ (we gave more details in the previous lecture). We consider the functor

$$\begin{aligned}\mathrm{Def}_L : \mathsf{Art}_k &\to \mathsf{Gpd} \\ R &\mapsto \mathrm{MC}(L \otimes m)/\exp(L^0 \otimes m).\end{aligned}$$

We saw in the last lecture that if A is an associative k-algebra, then the shifted Hochschild complex $C^\bullet(A)[1]$ forms a dg Lie algebra under the Gerstenhaber bracket. Moreover, if X is the functor that assigns to a ring S the groupoid of S-algebras, and take the point x of X corresponding to A, then there is an isomorphism of functors

$$\mathrm{Def}_{C^\bullet(A)[1]} = X_x.$$

This isomorphism indicates that the dg Lie algebra $C^\bullet(A)[1]$ controls the local structure of the moduli problem of associative algebras that the point A. Similarly, there are dg Lie algebras that control the deformations of Poisson algebras (via the complex of polyvector fields) and Lie algebras (via the Chevalley-Eilenberg complex).

Question: Is it true more generally that, given a point x of a functor $X : \mathsf{Ring} \to \mathsf{Gpd}$, we can find a dg Lie algebra L such that $X_x = \mathrm{Def}_L$?

The affirmative answer has been accepted as an experimental fact in many cases when X arises naturally from a geometric deformation problem. Our goal in the next sections is to give a precise formulation of this phenomenon in the setting of ∞-categories, where we can state a theorem that responds to a higher version of this question.

3 Higher algebra reminders

The first reminder is of simplicial localization, which is a major source of ∞-categories. Let $\mathcal{C}$ be a category and W a set of morphisms of $\mathcal{C}$. We can formally invert the morphisms in W to produce a category $\mathcal{C}[W^{-1}]$ called the Gabriel-Zisman localization of $\mathcal{C}$ with respect to W. The idea is that the objects of $\mathcal{C}[W^{-1}]$ are the same as the objects of $\mathcal{C}$, while the morphisms are given by zigzags, where the 'backward' pointing arrows are required to be in W. An example to keep in mind is the derived category $D(R)$ of a ring R as the localization of the category Chain_R of chain complexes over R along quasi-isomorphisms.

Gabriel-Zisman localization undesirable in general for several reasons. First, significant structure is lost when passing from $\mathcal{C}$ to $\mathcal{C}[W^{-1}]$. For example, if $\mathcal{C}$ is abelian, or if it had all limits and colimits, then $\mathcal{C}[W^{-1}]$ may not share these properties. Another shortcoming of Gabriel-Zisman localization is that, for more geometric categories, gluing constructions that work on the level of categories may not descend to the localized categories. The main example is the derived category of quasi-coherent sheaves on $\mathbb{P}^1$. So, despite its name, the localization $\mathcal{C}[W^{-1}]$ is not local in nature.

Instead, there is a construction known as simplicial (or Dwyer-Kan) localization that produces an ∞-category with the usual localization universal property in the homotopy category of ∞-categories. Henceforth, $\mathcal{C}[W^{-1}]$ will denote the simplicial localization rather than the Gabriel-Zisman localization. The details of the construction are a topic for another time. Roughly, the strategy is to produce a simplicial set out of the collection of zigzags that form the morphisms between any two objects in the Gabriel-Zisman localization. Since simplicially enriched categories are a model for ∞-categories the result can be regarded as an ∞-category.

For example, localizing the category chain complexes over a ring R along quasi-isomorphisms produces the so-called ∞-derived category $D^\infty(R)$ of R-modules. The homotopy category of $D^\infty(R)$ is the usual derived category $D(R)$ of R-modules.

Other examples that are relevant for our discussion are (1) the localization of the category $\mathsf{Lie}_k^{\mathrm{dg}}$ of dg Lie algebras over a field k along quasi-isomorphisms, and (2) the localization $\mathcal{S}$ of the category of CW complexes along weak homotopy equivalences. The latter is called the ∞-category of spaces.

The second reminder is that, in the setting of ∞-categories, classical algebraic structures are replaced by their homotopical analogues. Instead of the category Set of sets, we consider the ∞-category $\mathcal{S}$ of spaces. Instead of the symmetric monoidal category of abelian groups with tensor product and unit the integers $\mathbb{Z}$, we consider the symmetric monoidal ∞-category of spectra with smash product and unit the sphere spectrum. One can embed the category of abelian groups into the category of spectra via the construction of Eilenberg-Maclane spectra. The commutative ring objects in spectra are called E_∞-rings; these include ordinary commutative rings as discrete spectra. Let Mod_R denote the ∞-category of module spectra for an E_∞-ring R. If R is an ordinary ring, then $\mathsf{Mod}_R \simeq D^\infty(R)$.

An alternative approach to the rest of this talk is to work in the ∞-derived category $D^\infty(\mathbb{Z})$ of $\mathbb{Z}$ instead of the category of spectra. For settings where all fields under consideration have characteristic zero, we can even restrict our attention to $D^\infty(\mathbb{Q})$.

Let k be a field, regarded as a discrete spectrum. There are two common models for the ∞-category $\mathsf{Alg}_k^{(n)}$ of E_n-algebras over k. The first is as representations of the little n-cubes operad in the symmetric monoidal ∞-category Mod_k of k-module spectra. The second is as the iterated algebra objects in Mod_k:

$$\mathsf{Alg}_k^{(n)} = \mathsf{Alg}(\mathsf{Alg}(\dots \mathsf{Alg}(\mathsf{Mod}_k)\dots)).$$

Both models generalize from Mod_k to any symmetric monoidal ∞-category. The ∞-category $\mathsf{Alg}_k^{(1)}$ can be identified with the simplicial localization of the category of dg algebras over k along quasi-isomorphisms. Also, any E_∞-algebra over k can be be regarded as an E_n-algebra over k for any k.

4 Derived and formal moduli problems

We define a *classical moduli problem* to be a functor $X_0 : \mathsf{Ring} \to \mathsf{Gpd}$. In particular, any scheme or stack is a classical moduli problem. The appropriate analogue of a classical moduli problem in the ∞-category setting is called a *derived moduli problem*, and is defined as a functor

$$X : E_\infty\mathsf{Ring} \to \mathcal{S},$$

where $E_\infty\mathsf{Ring}$ denotes the category of E_∞-rings. We say that the derived moduli problem X is an enhancement of the classical moduli problem X_0 if the following diagram commutes:

$$\begin{array}{ccc} \mathsf{Ring} & \xrightarrow{X_0} & \mathsf{Gpd} \\ \downarrow & & \downarrow \\ E_\infty\mathsf{Ring} & \xrightarrow{X} & \mathcal{S} \end{array}$$

where the left vertical map is the inclusion of commutative rings as discrete E_∞-ring spectra, and the right vertical map is the inclusion of groupoids as 1-types.

In the classical setting, we studied the local structure of a moduli problem by choosing a k-rational point and probing that point with Artin k-algebras. We adopt a similar approach in studying the local structure of a derived moduli problem.

A *point* of a derived moduli problem X is a point $x = (k, \eta)$ where k is a field and η is a point of the topological space $X(k)$. Now we can again use Artin k-algebras to investigate the local structure of X at x. But one reason for passing to the homotopical setting is that we have more algebras (i.e ring spectra) that are 'close' to k, and we can use these to obtain a more complete picture of X near x. The analogue of an Artin k-algebra is an E_∞-*Artin* k-*algebra*, which is defined as a E_∞-algebra R over k that satisfies

- $\pi_0(R)$ is an (ordinary, commutative) Artin k-algebra.
- $\dim(\pi_n(R))$ is finite for all n and is 0 for $n < 0$ and $n >> 0$.

In particular, there is an augmentation map of E_∞-algebras $\epsilon : R \to k$. Let Art_k^∞ denote the category of E_∞-Artin k-algebras.

Given a derived moduli problem $X : E_\infty\mathsf{Ring} \to \mathcal{S}$ and a point $x = (k, \eta)$ of X, we have a functor

$$\begin{aligned} X_x : \mathsf{Art}_k^\infty &\to \mathcal{S} \\ R &\mapsto X(\epsilon)^{-1}(\eta) \end{aligned}$$

As in the classical case, the intuition is that X_x encodes local information about the moduli problem X. Under mild assumptions on X, the functor X_x is an example of a formal moduli problem. A *formal moduli problem over* k is a functor $F : \mathsf{Art}_k^\infty \to \mathcal{S}$ that satisfies the following two conditions:

- The space $F(k)$ is contractible.
- Suppose $A \to B$ and $A' \to B$ are morphisms in Art_k^∞ that are surjections on π_0. Then the induced map $F(A \times_B A') \to F(A) \times_{F(B)} F(A')$ is a homotopy equivalence.

One should think of the second condition as a gluing condition. Note that π_0 picks out the 'geometric' piece of a spectrum.

Recall that if X is a scheme and $x = (k, \eta)$ is a point, then the tangent space of X at x is the fiber over η under the map $X(k[\epsilon]/(\epsilon^2)) \to X(k)$. It is possible to perform an analogue of this construction in the setting of higher algebra. Let $F : \mathsf{Art}_k^\infty \to \mathcal{S}$ be an formal moduli problem. (The picture to have in mind is that F gives information about the local structure of a derived moduli problem $X : E_\infty\mathsf{Ring} \to \mathcal{S}$.)

Let $T_F(0) = F(k[\epsilon]/(\epsilon^2))$. Note that the ring $k[\epsilon]/(\epsilon^2)$ can be identified with the square-zero extension $k \oplus k$. If $k[n]$ denotes the n-fold shift of k (considered as a discrete spectrum) in the category of spectra, then we can similarly consider the square-zero extension $k \oplus k[n]$, which is an ∞-Artin k-algebra. Now let $T_F(n) = F(k \oplus k[n])$. One can show that $T_F = \{T_F(n)\}_{n>0}$ defines a spectrum, called the tangent complex of F.

(The reason T_F is called a complex is that there is an equivalence of the ∞-categories of k-module spectra and the simplicial localization $D^\infty(k)$ of the chain complexes of k-modules along quasi-isomorphisms.)

Claim. The spectrum $T_F[-1]$ has the structure of a dg Lie algebra over k.

The heuristic explanation is:

- Define $\Omega F : \mathsf{Art}_k^\infty \to \mathcal{S}$ by $(\Omega F)(R) = \Omega(F(R))$. In words, $(\Omega F)(R)$ is the loop space of $F(R)$.
- Argue that ΩF is a formal moduli problem and in fact a group object (or an $E_1 = A_\infty$ object) in the ∞-category of formal moduli problems.
- Therefore, $T_{\Omega F}$ has an dg Lie algebra (i.e. L_∞) structure.
- Show that $T_F[-1] = T_{\Omega F}$.

It turns out that F is determined by T_F in the following sense:

Theorem 1. *Assume that the characteristic of k is zero. Then there is an equivlance of ∞-categories*

$$\text{Moduli} \xrightarrow{\sim} \mathsf{Lie}_k^{\text{dg}}[W^{-1}]$$

given by $F \to T_F[-1]$, where Moduli $\subset$ Fun$(\mathsf{Art}_k^\infty, \mathcal{S})$ *denotes the full subcategory of formal moduli problems.*

Roughly, the inverse is given by considering solutions to the Maurer-Cartan equation of a dg Lie algebra and producing a formal moduli problem similar to the functor Def_L defined above. This theorem reflects the Koszul duality of the commutative algebra operad (appearing via Art_k^∞) and Lie algebra operad. It also justifies the thesis that the local structure of moduli space at a point is controlled by a dg Lie algebra (c.f. the question at the end of section 2 above).

5 Noncommutative formal moduli problems

Now we consider the noncommutative case by passing from E_∞-algebras to E_n-algebras over k. Once again, we are interested in the E_n-algebras over k that are 'close' to k, and we arrive that following generalization of what we called E_∞-Artin k-algebras: An E_n-algebra R over a field k is *E_n-Artin over k* if

- $\pi_0(R)/\text{nil}(\pi_0(R)) \simeq k$, so $\pi_0(R)$ is a *noncommutative* Artin k-algebra with residue field k.
- $\dim(\pi_n(R))$ is finite for all n and is 0 for $n < 0$ and $n >> 0$.

The category of small E_n-algebras over k is denoted Art_k^n. We define a *formal E_n-moduli problem over k* to be a functor $F : \mathsf{Art}_k^n \to \mathcal{S}$ such that

- The space $F(k)$ is contractible.
- Suppose $A \to B$ and $A' \to B$ are morphisms in Art_k^n that are surjections on π_0. Then the induced map $F(A \times_B A') \to F(A) \times_{F(B)} F(A')$ is a homotopy equivalence.

Each E_∞-Artin k-algebra is in particular and E_n-Artin k-algebra, that is, there is an inclusion $\iota : \mathsf{Art}_k^\infty \to \mathsf{Art}_k^n$. So any formal E_n-moduli problem F defines a formal moduli problem, and the tangent complex is defined as the tangent complex of $F \circ \iota$.

Theorem 2. *Let k be a field (of any characteristic). There is an equivalence of ∞-categories*

$$\Phi : \mathrm{Moduli}_n \xrightarrow{\sim} \mathsf{Alg}^{(n)}_{\mathrm{aug}}$$

where $\mathrm{Moduli}_n \subset \mathrm{Fun}(\mathsf{Art}^n_k, \mathcal{S})$ *denotes the full subcategory of formal E_n-moduli problems over k and* $\mathsf{Alg}^{(n)}_{\mathrm{aug}}$ *denotes the category of augmented E_n-algebras over k.*

It turns out that the augmentation ideal of the image $\Phi(F)$ can be identified with the shifted tangent complex $T_F[-n]$ of F. This theorem reflects the Koszul self-duality of the E_n operad, for $n \geq 0$. In Chain_k, representations of the E_n operad for $n \geq 2$ are vector spaces equipped with a commutative multiplication and a Lie bracket of degree $n-1$. In this sense, a representation of E_n is a combination of representations of the operads Comm and Lie. If we accept that the operads Comm and Lie are mutually dual, then the above observation gives one explanation for the self-duality of the E_n operad.