Associators, Talk 1

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0 Introduction

The goal of this talk is to introduce (1) the notion of a quantum universal enveloping quasi-bialgebra (QUE algebra for short) for a Lie algebra, (2) Drinfeld's associators as a way to construct QUE algebras, and (3) the Grothendieck-Teichmüller group as the symmetries associators for braided monoidal categories. Section 1 gives background on quasi-bialgebras. In Section 2, we define QUE algebras as quantizations of Casimir Lie algebras. We also define the holonomy Lie algebra $\mathfrak{t}_n$ in order to introduce the concept of a Drinfeld associator. Section 3 explains how the process of changing the associativity and commutativity constraints on a braided monoidal category, which maintaining the rest of the structure, leads to the definition of (the pro-nilpotent version of) the Grothendieck-Teichmüller group.

1 Quasi-bialgebras

This section introduces background on quasi-bialgebras. Quasi-bialgebras are generalizations of bialgebras in which the coproduct is only associative up to a 'coassociator'. As we will see, an advantage of quasi-bialgebras is that they admit certain symmetries, called gauge transformations, making them more flexible to work with than ordinary bialgebras. We assume familiarity with bialgebras, Hopf algebras, universal R-matrices, braided monoidal categories, and rigid monoidal categories. The proofs given here are more precisely only sketches of proofs; the audience is invited to fill in the details.

1.1 Definition of quasi-bialgebras

Let k be a field. Let A be an associative unital algebra over k. Fix homomorphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$. Observe that if U, V are A-modules, then the tensor product $U \otimes V$ carries an action of A given by the composition

$$A \xrightarrow{\Delta} A \otimes A \to \mathrm{End}(U) \otimes \mathrm{End}(V) \to \mathrm{End}(U \otimes V).$$

Also, the field k becomes an A-module via the map ϵ. Let $A{-}\mathsf{mod}$ denote the category of A-modules.

Question: when does this tensor product (with unit given by ϵ) define a monoidal structure on $A{-}\mathsf{mod}$?

In other words, we need A-linear asociativity isomorphisms $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ that satisfy the pentagon and triangle axioms. Recall that, if (A, Δ, ϵ) is a bialgebra (in particular, Δ is coassociative), then $A{-}\mathsf{mod}$ is monoidal with the trivial associator $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w)$.

Idea: A weaker version of coassociativity for Δ can still be used to define a monoidal structure, and leads to the notion of a quasi-bialgebra.

Definition. A *quasi-bialgebra* is an algebra A over k together with homomorphisms $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$, and an invertible element $\Phi \in A \otimes A \otimes A$ that satisfy the following equations:

$$((\mathrm{Id} \otimes \Delta) \circ \Delta)(a) = \Phi^{-1}.((\Delta \otimes \mathrm{Id}) \circ \Delta)(a).\Phi, \quad \text{for all } a \in A \tag{1}$$

$$(\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})(\Phi).(\mathrm{Id} \otimes \mathrm{Id} \otimes \Delta)(\Phi) = (\Phi \otimes \mathrm{Id}).(\mathrm{Id} \otimes \Delta \otimes \mathrm{Id})(\Phi).(\mathrm{Id} \otimes \Phi) \tag{2}$$

$$(\epsilon \otimes \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes \epsilon) \circ \Delta \tag{3}$$

$$(\mathrm{Id} \otimes \epsilon \otimes \mathrm{Id})(\Phi) = 1. \tag{4}$$

Remark. The element Φ is often called a 'coassociator'. Any bialgebra is a quasi-bialgebra with the trivial coassociator $\Phi = 1 \otimes 1 \otimes 1$.

Proposition 1. *Let A be a quasi-bialgebra. Then $A{-}\mathsf{mod}$ is a monoidal category with the above tensor product, unit object $\epsilon : A \to k$, and associativity isomorphisms given by the (componentwise) action of Φ:*

$$\begin{aligned} a_{U,V,W} : (U \otimes V) \otimes W &\to U \otimes (V \otimes W) \\ (u \otimes v) \otimes w &\mapsto \Phi(u \otimes (v \otimes w)). \end{aligned}$$

Proof. Equation (1) implies the maps $\alpha_{U,V,W}$ are A-modules homomorphisms and equation (2) ensures that they satisfy the pentagon axiom. Equation (3) implies that $\epsilon : A \to k$ is the unit object and equation (4) implies the triangle axiom. □

Remark. There is a notion of a quasi-bialgebra where the unit constraints are loosened, but it seems to occur more seldomly in practice, so we omit it here.

1.2 Quasi-fiber functors

In this subsection, we will see that so-called quasi-fiber functors provide a source of quasi-bialgebras.

Definition. A *monoidal functor* between monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a pair (F, J) consisting of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural isomorphism $J : F(-) \otimes F(-) \to F(- \otimes -)$, called the *monoidal structure* on F such that:

1. J is compatible with the associativity constraints, i.e. the following diagram commutes:

$$\begin{tikzcd} (F(U) \otimes F(V)) \otimes F(W) \arrow[r, "a^{\mathcal{D}}_{F(U),F(V),F(W)}"] \arrow[d, "J \otimes \mathrm{Id}"] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d, "\mathrm{Id} \otimes J"] \\ F(U \otimes V) \otimes F(W) \arrow[d, "J"] & F(U) \otimes F(V \otimes W) \arrow[d, "J"] \\ F((U \otimes V) \otimes W) \arrow[r, "F(a^{\mathcal{C}}_{U,V,W})"] & F(U \otimes (V \otimes W)) \end{tikzcd} \tag{5}$$

   Here, $a^{\mathcal{C}}$ and $a^{\mathcal{D}}$ are the associativity constraints in $\mathcal{C}$ and $\mathcal{D}$, respectively.

2. $F(1_{\mathcal{C}}) = 1_{\mathcal{D}}$

3. J is compatible with the unit constraints (see Kassel, XI.4.1, diagrams (4.2) and (4.3)).

A *quasi-monoidal functor* is a pair (F, J) as above that satisfies items 2 and 3, but not necessarily item 1. In this case, J is called a *quasi-monoidal structure* on F. We say that a natural transformation $\eta : F \to F$ is *compatible* with a quasi-monoidal structure J if $J_{U,V} \circ \eta_U \otimes \eta_V = \eta_{U \otimes V} \circ J_{U,V}$ for all objects U, V in $\mathcal{C}$.

Remark. A weaker version of the above definition is to require the data of an isomorphism $F(1_{\mathcal{C}}) \xrightarrow{\sim} 1_{\mathcal{D}}$, and the appropriate modifications to the compatiblility with the unit constraints.

Definition. A *(quasi-)fiber functor* is a faithful and exact (quasi-)monoidal functor $(F, J) : \mathcal{C} \to \mathrm{Vec}_k^{\mathrm{fd}}$ from a k-linear monoidal category $\mathcal{C}$ the the category of finite-dimensional vector spaces over k.

Let $A{-}\mathsf{mod}^{\mathrm{fd}}$ denote the category of A-modules that are finite-dimensional as k-vector spaces.

Proposition 2. *Let $(F, J) : \mathcal{C} \to \mathrm{Vec}_k^{fd}$ be a (quasi-)fiber functor. The algebra $A = \mathrm{End}^J(F)$ of endomorphisms of F compatible with J naturally forms a (quasi-)bialgebra. There is a natural functor $\mathcal{C} \to A{-}\mathsf{mod}^{fd}$ which, under finiteness assumptions on $\mathcal{C}$, is an equivalence of categories.*

Proof. Abbreviate $\mathrm{End}^J(F)$ by $\mathrm{End}(F)$. The algebra structure on $A = \mathrm{End}(F)$ is given by the composition of natural transformations. The comultiplication is defined as follows. First, since natural transformations pull back under functors, there is a map $A = \mathrm{End}(F) \to \mathrm{End}(F \circ \otimes)$. Next, the natural isomorphism J defines an isomorphism $\mathrm{End}(F \circ \otimes) \simeq \mathrm{End}(\otimes \circ (F \times F))$.

$$\begin{tikzcd}
 & \mathcal{C} \times \mathcal{C} \arrow[dl, "F \times F"'] \arrow[dr, "\otimes"] & \\
\mathrm{Vec}_k^{\mathrm{fd}} \times \mathrm{Vec}_k^{\mathrm{fd}} \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{J}{\Leftarrow}"] & & \mathcal{C} \arrow[dl, "F"] \\
 & \mathrm{Vec}_k^{\mathrm{fd}} &
\end{tikzcd}$$

Now, $\mathrm{End}(\otimes \circ (F \times F)) \simeq \mathrm{End}(F) \otimes \mathrm{End}(F) = A \otimes A$. Putting these pieces together, we obtain a map $\Delta : A \to A \otimes A$. The counit is defined as the map $\epsilon : A = \mathrm{End}(F) \to k$ taking a to $a_{F(1)} = a_k \in \mathrm{End}_k(k) = k$. The coassociator Φ on $A = \mathrm{End}^J(F)$ emerges from the possible failure of the commutativity of diagram 5. $\square$

1.3 Quasitriangular quasi-bialgebras

Recall that a quasitriangular bialgebra is a bialgebra A with the extra structure of a universal R-matrix. The universal R-matrix endows $A-\mathsf{mod}$ with the structure of a braided monoidal category. Similarly, there is a notion of a universal R-matrix in the setting of quasi-bialgebras.

Definition. A *quasitriangular quasi-bialgebra* is a quasi-bialgebra together with an invertible element $\mathcal{R} \in A \otimes A$, called its *universal R-matrix*, that satisfies

$$\Delta^{\mathrm{op}}(a) = \mathcal{R}.\Delta(a).\mathcal{R}^{-1} \tag{6}$$

$$(\Delta \otimes \mathrm{Id})(\mathcal{R}) = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{132}\mathcal{R}_{23}\Phi_{123}^{-1} \tag{7}$$

$$(\mathrm{Id} \otimes \Delta)(\mathcal{R}) = \Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123} \tag{8}$$

If, in addition, $\mathcal{R}$ satisfies $\mathcal{R}_{21} = \mathcal{R}_{12}^{-1}$ and $(\epsilon \otimes \epsilon)\mathcal{R} = 1$, then we say that $(A, \mathcal{R})$ is a *triangular quasi-bialgebra.*

Here, if $\Phi = \sum_s \phi_s \otimes \phi'_s \otimes \phi''_s$, then define $\Phi_{312} = \sum_s \phi''_s \otimes \phi_s \otimes \phi'_s$, etc.

Proposition 3. *Let A be a quasitriangular (resp. triangular) quasi-bialgebra. Then $A-\mathsf{mod}$ has the natural structure of a braided (resp. symmetric) monoidal category.*

Proof. For $U, V \in A-\mathsf{mod}$, define

$$\begin{aligned} c_{U,V} &: U \otimes V \to V \otimes U \\ & u \otimes v \mapsto (12) \circ (\rho_U \otimes \rho_V)(\mathcal{R})(u \otimes v) \end{aligned}$$

where $(12) : U \otimes V \to V \otimes U$ is the usual switch map $u \otimes v \mapsto v \otimes u$, and $\rho_U : A \to \mathrm{End}(U)$ and $\rho_V : A \to \mathrm{End}(V)$ are the action maps, so that $\rho_U \otimes \rho_V$ defines a map $A \otimes A \to \mathrm{End}(U) \otimes \mathrm{End}(V) \simeq \mathrm{End}(U \otimes V)$. Then equation (6) implies that $c_{U,V}$ is A-linear and equations (7) and (8) guarantee that the hexagon axioms hold. $\square$

Remark. A universal R-matrix $\mathcal{R}$ satisfies the so-called 'quasi-quantum Yang Baxter equation':

$$\mathcal{R}_{12}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}\Phi_{123} = \Phi_{321}\mathcal{R}_{23}\Phi_{312}\mathcal{R}_{13}\Phi_{213}^{-1}\mathcal{R}_{12}.$$

1.4 Gauge transformations

One advantage of quasi-bialgebras over ordinary algebras is the existence of certain symmetries on the category of quasi-bialgebras.

Definition. Let $(A, \Delta, \epsilon, \Phi)$ be a quasi-bialgebra. A *gauge transformation* is an invertible element $F \in A \otimes A$ such that

$$(\epsilon \otimes \mathrm{Id})(F) = 1 = (\mathrm{Id} \otimes \epsilon)(F).$$

For such an F, define:

$$\Delta^F(a) = F.\Delta(a).F^{-1}, \qquad \Phi^F = F_{12}.(\Delta \otimes \mathrm{Id})(F).\Phi.(\mathrm{Id} \otimes \Delta)(F)^{-1}.F_{23}^{-1}.$$

If A is quasitriangular with universal R matrix $\mathcal{R}$, define:

$$\mathcal{R}^F = F_{21}.\mathcal{R}.F_{12}^{-1}.$$

Lemma 4. *Let A be a quasi-bialgebra and let $F \in A \otimes A$ be a gauge transformation. Then $(A, \Delta^F, \epsilon, \Phi^F)$ is a quasi-bialgebra, denoted A^F. If A is quasitriangular with universal R-matrix $\mathcal{R}$, then $\mathcal{R}^F$ is a universal R-matrix for A^F.*

Definition. Let A and B be two (quasitriangular) quasi-bialgebras. If $B = A^F$ for some $F \in A \otimes A$ as above, then we say that A and B are *twist equivalent* or *gauge equivalent* and that B is obtained from A by twisting the (quasitriangular) quasi-bialgebra structure by F.

Note that twisting forms an equivalence relation on quasi-bialgebras. In fact, this equivalence extends to the categorical level:

Proposition 5. *Let A be a quasi-bialgebra. For any F as above, the categories $A-\mathsf{mod}$ and $A^F-\mathsf{mod}$ are equivalent as braided monoidal categories. If, moreover, A is quasitriangular, then $A-\mathsf{mod}$ and $A^F-\mathsf{mod}$ are equivalent as braided monoidal categories.*

Proof. To define an equivalence of monoidal categories, we must construct a functor $f : A-\mathsf{mod} \to A^F-\mathsf{mod}$ as well as a natural isomorphism $J : f(- \otimes -) \to f(-) \otimes f(-)$ that satisfies certain axioms related to the monoidal structure. Take f to be the identity, noting that A and A^F are identical as algebras. For U, V in $A-\mathsf{mod}$, take $J_{U,V}$ to the be image of F in $\mathrm{End}(U \otimes V)$. The proof reveals that Δ^F, Φ^F, and $\mathcal{R}^F$ were defined to ensure the existence of this equivalence. ☐

Corollary 6. *There is a bijective correspondence between gauge transformations on a quasi-bialgebra A and quasi-monoidal structures J on the forgetful functor $F : A-\mathsf{mod} \to \mathrm{Vec}_k$.*

Proof. Given a gauge transformation F, define $J_{U,V}$ using the action of F on $U \otimes V$. Given a quasi-monoidal structure J, define F as the image of $1 \otimes 1$ under $J_{A,A}$. ☐

1.5 Quasi-Hopf algebras

Recall that a Hopf algebra is a bialgebra A over k together with an antihomomorphism $S : A \to A$, called the antipode, that satisfies the antipode axiom:

$$\sum_r S(a_r)a'_r = \epsilon(a)$$

for all $a \in A$, where $\Delta(a) = \sum_r a_r \otimes a'_r$. If U is an object in the category $A-\mathsf{mod}^{\mathrm{fd}}$ of finite-dimensional modules for a Hopf algebra A, then A acts on the dual $U^* = \mathrm{Hom}_k(U, k)$ by $\langle a.\xi, u\rangle = \langle \xi, S(a).u\rangle$. The antipode axiom ensure that the evaluation map $U^* \otimes U \to k$ and the coevaluation map $k \to U \otimes U^*$ are A-linear. Moreover, the evaluation and coevaluation maps satisfy the rigidity axioms; consequently, $A-\mathsf{mod}^{\mathrm{fd}}$ has the structure of a *rigid* monoidal category. Recall also that if an antipode exists, then it is unique.

Previously we investigated a weaker version of coassociativity of a bialgebra that still produced the monoidal structure on the category of modules. Similarly, a weakening of both the anitpode axiom and coassociativity leads to a generalization of a Hopf algebra, called a quasi-Hopf algebra, whose category of modules is a rigid monoidal category.

Definition. A *quasi-Hopf algebra* is a quasi bialgebra $(A, \Delta, \epsilon, \Phi)$ together with an antihomomorphism $S : A \to A$ and elements $\alpha, \beta \in A$ that satisfy

$$\sum_r S(a_r)\alpha a'_r = \epsilon(a)\alpha, \qquad \sum_r a_r \beta S(a'_r) = \epsilon(a)\beta \tag{9}$$

for all $a \in A$, and

$$\sum_s S(\phi_s)\alpha\phi'_s\beta S(\phi''_s) = 1, \qquad \sum_t \bar{\phi}_t\beta S(\bar{\phi}'_t)\alpha\bar{\phi}''_t = 1. \tag{10}$$

Here

$$\Delta(a) = \sum_r a_r \otimes a'_r, \qquad \Phi = \sum_s \phi_s \otimes \phi'_s \otimes \phi''_s, \quad \Phi^{-1} = \sum_t \bar{\phi}_t \otimes \bar{\phi}'_t \otimes \bar{\phi}''_t.$$

A universal R-matrix for a quasi-Hopf algebra is a universal R-matrix for the underlying quasi-bialgebra. A *quasitriangular quasi-Hopf algebra* is a quasi-Hopf algebra equipped with a universal R-matrix.

Observe that any Hopf algebra is a quasi-Hopf algebra with $\Phi = 1 \otimes 1 \otimes 1$ and $\alpha = \beta = 1$. The following proposition follows easily by checking the rigidity axioms.

Proposition 7. *Let $(A, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra. The category $A-\mathsf{mod}^{fd}$ is a rigid monoidal category, where the evaluation and coevaluation maps are given by*

$$U^* \otimes U \to k \qquad\qquad k \to U \otimes U^* \tag{11}$$

$$\xi \otimes u \mapsto \langle \xi, a.u \rangle \qquad\qquad 1 \mapsto \sum_i \beta.u_i \otimes \xi^i \tag{12}$$

A gauge transformation of a quasi-Hopf algebra is a gauge transformation of the underlying quasi-bialgebra, together with a twisting of the elements α and β.

Definition. Let $(A, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra and let $F = \sum_i f_i \otimes g_i$ be a gauge transformation on A with inverse $\sum_j \bar{f}_j \otimes g_j$. Define

$$\alpha^F = \sum_j S(\bar{f}_j)\alpha\bar{g}_j, \qquad \beta^F = \sum_i f_i\beta S(g_i).$$

Proposition 8. *Let $(A, \Delta, \epsilon, \Phi, S, \alpha, \beta)$ be a quasi-Hopf algebra and F a gauge transformation on A. Then $(A, \Delta^F, \epsilon, \Phi^F, S, \alpha^F, \beta^F)$ is a quasi-Hopf algebra.*

Example. Let G be a finite group. Let $\mathbb{C}[G]$ be the group algebra of G, and let $\mathcal{O}(G)$ be the algebra of complex-valued functions on G. Let $\mathcal{D}(G) = \mathcal{O}(G) \otimes \mathbb{C}[G]$. Given a (normalized) 3-cocyle $c : G \times G \times G \to S^1$ with values in the circle group, one can define a quasi-Hopf algebra structure on $\mathcal{D}(G)$, denoted $\mathcal{D}^c(G)$. It turns out that $\mathcal{D}^c(G)$ and $\mathcal{D}^{c'}(G)$ are twist-equivalent if and only if the cocycles c and c' are cohomologous. Hence, up to twist equivalence, the quasi-Hopf algebras $\mathcal{D}^c(G)$ are classified by $H^3(G, S^1)$. This example is due to Dijkgraaf, Pasquier, and Roche, and relates to conformal field theories with symmetry group G.

2 QUE algebras

Observe that all constructions of the previous section make sense for algebras over the ring $k[[h]]$ of formal power series in k, where all tensor products are replaced by completed tensor products. One can show that, if A is a topological (quasitriangular) quasi-bialgebra, then A/hA inherits the structure of a (quasitriangular) quasi-bialgebra. See Kassel, Section XVI.4 for more details.

2.1 Definition of QUE algebras

We work over $\mathbb{C}$. Set $K = \mathbb{C}[[h]]$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$, so $U(\mathfrak{g})$ is a bialgebra with the usual coproduct $\Delta(x) = 1 \otimes x + x \otimes 1$ and counit $\epsilon(x) = 1$ for $x \in \mathfrak{g}$. Set $R_{U(\mathfrak{g})} = 1 \otimes 1$ and $\Phi_{U(\mathfrak{g})} = 1 \otimes 1 \otimes 1$ making $U(\mathfrak{g})$ into a quasitriangular quasi-bialgebra.

Definition. A *quantum universal enveloping* (QUE) algebra for $\mathfrak{g}$ is a topological quasitriangular quasi-bialgebra $A = (A, \Delta, \epsilon, \Phi, R)$ such that A is topologically free as a K-module and A/hA is isomorphic to $U(\mathfrak{g})$ as a quasitriangular quasi-bialgebra.

Let A be a QUE algebra. Observe that:

- $A \simeq U(\mathfrak{g})[[h]]$ as K-modules.
- The coproduct Δ and counit ϵ on A are determined by the two conditions on A.
- We have that $\Phi \equiv 1 \otimes 1 \otimes 1$ and $R \equiv 1 \otimes 1 \mod h$.
- The Lie algebra $\mathfrak{g}$ is an invariant of A, obtained as the primitive elements of A/hA:

$$\mathfrak{g} = \{x \in A/hA \mid \Delta_0 = 1 \otimes x + x \otimes 1\}.$$

2.2 Casimir Lie algebras and quantization

We introduce another invariant of a QUE A:

Definition. Let A be a QUE algebra for $\mathfrak{g}$. The *cannonical 2-tensor* of A is the element $t \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by
$$R_{21}R \equiv 1 \otimes 1 + ht \mod h^2.$$

Proposition 9. *Let A be a QUE algebra for $\mathfrak{g}$ and $t \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ the cannonical 2-tensor. Then t is a $\mathfrak{g}$-invariant symmetric element of $\mathfrak{g} \otimes \mathfrak{g}$ and is invariant under gauge transformations.*

Definition. A *Casimir Lie algebra* is a pair $(\mathfrak{g}, t)$ consisting of a Lie algebra $\mathfrak{g}$ together with a $\mathfrak{g}$-invariant symmetric 2-tensor $t \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$. If t is the cannonical element of a QUE algebra A for $\mathfrak{g}$, then $(\mathfrak{g}, t)$ is called the *classical limit* of A and A is called a *quantization* of $(\mathfrak{g}, t)$.

Diagrammitcally:

$$\begin{tikzcd} \text{QUE algebras} \arrow[r, "\text{classical limit}"'] & \text{Casimir Lie algebras } (\mathfrak{g}, t) \arrow[l, bend right, dotted, "\text{quantization}"'] \end{tikzcd}$$

Example. In $\mathfrak{sl}_2$, we take $t = 1/4(E \otimes F + F \otimes E + (H \otimes H)/2)$ coming from the Killing form (see Kassel, XVII.1). The Drinfeld-Jimbo quantum group for $\mathfrak{sl}_2$ is a quantization of the corresponding Casismir Lie algebra.

Question: Given a Casimir Lie algebra $(\mathfrak{g}, t)$, does a quantization exist?

We know that the quantization A is isomorphic to $U(\mathfrak{g})[[h]]$ as an algebra. To guarantee that $R_{21}R \equiv 1 + ht \mod h^2$, we set $R = e^{ht/2}$. Now the question becomes: can we find an associator Φ that works?

Theorem 10 ([2])**.** *There is a Φ that quantizes the Casimir Lie algebra $(\mathfrak{g}, t)$. It is unique up to to gauge transformations $F \in \mathrm{Sym}^2(A)^{\mathfrak{g}}$.*

2.3 The holonomy Lie algebra

The strategy for finding Φ relies on Kohno's holonomy Lie algebra. Although it is not crucial to the main ideas of this talk, this Lie algebra will appear in later talks.

Definition. Fix a positive integer n. The *holonomy Lie algebra* $\mathfrak{t}_n$ is the Lie algebra over $\mathbb{C}$ generated by X^{ij} for $1 \le i, j \le n$, $i \ne j$ with the following relations:

- $X^{ij} = X^{ji}$
- $[X^{ij}, X^{kl}] = 0$ for distinct i, j, k, l
- $[X^{ij} + X^{ik}, X^{jk}] = 0$ for distinct i, j, k

These relations are known as the *infinitesimal braid relations.* The holonomy Lie algebra can be realized as follows. Let $(\mathfrak{g}, t)$ be a Casimir Lie algebra, where $\mathfrak{g}$ is simple. Write $t = \sum a \otimes b$ and define

$$t^{ij} = \sum 1 \otimes \ldots \otimes 1 \otimes a \otimes 1 \otimes \ldots \otimes 1 \otimes b \otimes 1 \otimes \ldots 1 \in U(\mathfrak{g})^{\otimes n},$$

where a appears in the ith slot and b appears in the jth slot. Then the t^{ij} satisfy the inifinitesimal braid relations, so $X^{ij} \mapsto t^{ij}$ is a representation of $\mathfrak{t}_n$.

2.4 Drinfeld associators

Let $\mathbb{C}\langle\langle x, y\rangle\rangle$ be the completed free algebra on 2 generators. There is a bialgebra structure on $\mathbb{C}\langle\langle x, y\rangle\rangle$ given by $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\Delta(y) = 1 \otimes y + y \otimes 1$. If ϕ is an element of $\mathbb{C}\langle\langle x, y\rangle\rangle$ and a, b are elements of a complete topological algebra A, then $\phi(a, b)$ is defined as the image of ϕ under the unique homomorphism $\mathbb{C}\langle\langle x, y\rangle\rangle \to S$ taking x to a and y to b.

Definition. Let $\lambda \in \mathbb{C}^\times$. An element $\phi \in \mathbb{C}\langle\langle x, y\rangle\rangle$ is called a *Drinfeld λ-associator* if

- ϕ is group-like, i.e. $\Delta(\phi) = \phi \otimes \phi$, and hence ϕ is invertible;
- $\phi(y, x) = \phi(x, y)^{-1}$ in $\mathbb{C}\langle\langle x, y\rangle\rangle$;
- (pentagon equation) the following identity holds in $\widehat{U(\mathfrak{t}_4)}$:

$$\phi(X^{12}, X^{23} + X^{24})\phi(X^{13} + X^{23}, X^{34}) = \phi(X^{23}, X^{34})\phi(X^{12} + X^{13}, X^{24} + X^{34})\phi(X^{12}, X^{23});$$

- (hexagon equation) the following identity holds in $\mathbb{C}\langle\langle x, y\rangle\rangle$, where $z = -x - y$:

$$\exp(\lambda x/2)\phi(z, x)\exp(\lambda z/2)\phi(y, z)\exp(\lambda y/2)\phi(x, y) = 1.$$

Let $\mathcal{S}$ denote the set of all λ-associators, for $\lambda \in \mathbb{C}^\times$.

Let $(\mathfrak{g}, t)$ be a Casimir Lie algebra, let A be $U(\mathfrak{g})[[h]]$ as a complete topological algebra with coproduct Δ and counit ϵ extended from $U(\mathfrak{g})$. Define $t_{12}, t_{23} \in \mathfrak{g} \otimes \mathfrak{g} \otimes g$ in the obvious way.

Theorem 11. *Suppose ϕ is a Drinfeld λ-associator. Let $(\mathfrak{g}, t)$ be a Casimir Lie algebra. Then*

$$(U(\mathfrak{g})[[h]], \Delta, \epsilon, \Phi = \phi(ht_{12}, ht_{23}), R = e^{h\lambda t/2})$$

is a QUE algebra for $(\mathfrak{g}, t)$.

Remark. Drinfeld constructed an explcit associator using the KZ equations (Pavel's talk).

3 Grothendieck-Teichmüller group

This section introduces the Grothendieck-Teichmüller group. Due to time constraints, this section has not been written as carefully as the previous sections.

3.1 Motivation and notation

Let $\mathcal{C}$ be a braided monoidal category. Thus, we have associativity and commutativity constraints

$$a_{U,V,W} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W, \qquad c_{U.V} : U \otimes V \to V \otimes U.$$

We consider the problem of changing the commutativity and associativity isomorphisms without changing the rest of the structure of the category. A meaningful way to think about this problem is in terms of the pure braid group.

Recall that the braid group B_n on n strands has a presentation:

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \text{Braid relations} : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \text{ etc.}\rangle.$$

There is a surjective map from the braid group to the symmetric group $B_n \to S_n$ taking the generator σ_i to the permutation $(i, i+1)$. The *pure braid group* K_n is defined as the kernel of this map $B_n \to S_n$. Pictorially, these are braids in which each strand begins and ends at the same node.

The pure braid group K_3 acts on $U \otimes (V \otimes W)$, for any three objects U, V, W of $\mathcal{C}$. Thus, if $k \in K_3$, then we define $a \circ k$ to be the new potential associativity constraint obtained from a by precomposing with the action of k. Similarly, K_2 acts on $U \otimes V$ and precomposing the commutativity isomorphism c with the action of $k' \in K_2$, we obtain a new potential commutativity constraint, denoted $c \circ k'$.

[Picture defining $a \circ k$ and $c \circ k'$.]

Although the maps $a \circ k$ and $c \circ k'$ are functorial, they are only *potential* associativity and commutativity constraints since they may not satisfy the pentagon and hexagon equations. We return to this point below, after introducing some notation:

- The groups B_2 and K_2 are free on one generator: $B_n = \langle \sigma \rangle \simeq \mathbb{Z}$ and $K_2 = \langle \sigma^2 \rangle \simeq \mathbb{Z}$.
- If f is an element of the free group $F_2 = \langle a, b \rangle$ on two letters and x, y are elements of a group G, then $f(x, y)$ is defined as the image of f under the unique homomorphism $F_2 \to G$ taking a to x and b to y.
- Every element of the group K_3 can be written as $f(\sigma_1^2, \sigma_2^2)(\sigma_1\sigma_2)^{3n}$ where $n \in \mathbb{Z}$ and $f \in F_2$.
- The group K_4 is generated by the elements

  $$x_{12} = \sigma_1^2 \qquad x_{13} = \sigma_2\sigma_1^2\sigma_2^{-1} \qquad x_{23} = \sigma_2^2 \qquad x_{24} = \sigma_3\sigma_2^2\sigma_3^{-1} \qquad x_{34} = \sigma_3^2.$$

  The relations can be written down explicitly, but we do not need them.

3.2 Discrete version of GT

Given a braided monoidal category $\mathcal{C}$ with associativity and commutativity constraints a and c, we have maps

$$\mathbb{Z} \times F_2 \times \mathbb{Z} \longrightarrow K_2 \times K_3 \longrightarrow \{\text{potentially new assoc. and comm. constraints}\}$$

where the first arrow is the surjection taking (m, f, n) to $(\sigma^{2m}, f(\sigma_1^2, \sigma_2^2)(\sigma_1\sigma_2)^{3n})$ and the second map takes (k, k') to the associativity constraint $a \circ k'$ and the commutivity constraint $c \circ k$.

Of course, the new potential associativity and commutativity constraints may not satisfy the hexagon and pentagon axioms, and thus may not define a braided monoidal structure on the underlying k-linear category of $\mathcal{C}$. One can show that the only new potential associativity and commutativity constraints are those in the image of the subset of $\mathbb{Z} \times F_2 \times \mathbb{Z}$ consisting of triples (m, f, n) where $n = 0$ and m, f satisfy the following:

$$f(X, Y) = f(Y, X)^{-1} \text{ for any } X, Y \in F_2 \tag{13}$$

$$f(X_3, X_1)X_3^m f(X_2, X_3)X_2^m f(X_1, X_2)X_1^m = 1 \text{ for } X_1X_2X_3 = 1 \text{ in } F_2 \tag{14}$$

$$f(x_{12}, x_{23}x_{24})f(x_{13}x_{23}, x_{34}) = f(x_{23}, x_{34})f(x_{12}x_{13}, x_{24}x_{34})f(x_{12}, x_{23}) \text{ in } K_4 \tag{15}$$

Let

$$\mathrm{GT}_{\mathrm{disc}} = \{(\lambda, f) \in \lambda \in 1 + 2\mathbb{Z} \times F_2 \mid m = (\lambda - 1)/2 \text{ and } f \text{ satisfy the equations above}\}.$$

Thus, every pair $(\lambda, f) \in \mathrm{GT}_{\mathrm{disc}}$ defines a functorial way to construct from any braided monoidal category $\mathcal{C}$ a new braided monoidal category $\mathcal{C}'$ where the only change is the associativity and commutativity isomorphisms. In fact, any such construction arises from an element of $\mathrm{GT}_{\mathrm{disc}}$. Interpreting $\mathrm{GT}_{\mathrm{disc}}$ in such a way endows it with a monoid structure given by

$$(\lambda_1, f_1) \cdot (\lambda_2, f_2) = (\lambda_1\lambda_2, f_1(f_2(X, Y)X^{\lambda_2}f_2(X, Y)^{-1}, Y^{\lambda_2}) \cdot f_2(X, Y)).$$

We call $\mathrm{GT}_{\mathrm{disc}}$ the *discrete Grothendieck-Teichmüller semigroup.* Unfortunately, we have:

Lemma 12. $\mathrm{GT}_{disc} = \{0, 1\}, \{-1, 1\}$.

3.3 Pro-nilpotent version of GT

More interesting things happen when things are done topologically and pro-nilpotently. First, observe that when $\mathcal{C}$ is the category of modules for a quasitriangular quasi-bialgebra $(A, \Delta, \epsilon, \Phi, R)$, then applying (λ, f) we obtain the category of modules for $(A, \Delta, \epsilon, \bar{\Phi}, \bar{R})$, where

$$\bar{R} = R.(R^{21}R)^m = (RR^{21})^m.R \qquad \bar{\Phi} = \Phi.f(R^{21}R^{12}, \Phi^{-1}R^{32}R^{23}\Phi).$$

If the characteristic of k is 0, then equations (13)-(15) still make sense if we replace λ with any element of k and f with any element in the pro-nilpotent completion F_2^{nil}. Then $f(X, Y)$ is a formal completion of the form $\exp(F(\ln X, \ln Y))$ where F is a Lie formal series over k.

We obtain a semigroup $\underline{\mathrm{GT}}(k)$ called the *k-pro-nilpotent version of the Grothendieck-Teichmüller group*. Let $\mathrm{GT}(k) \subseteq \underline{\mathrm{GT}}(k)$ be the group of invertible elements. Then $\mathrm{GT}(k)$ acts on the category topological quasitriangular quasi-bialgebras over $k[[h]]$.

The group $\mathrm{GT}(\mathbb{C})$ acts on the set $\mathcal{S}$ of Drinfeld associators as

$$(f, \lambda) \cdot (\phi, \mu) = (f(\phi e^A \phi^{-1}, e^B)\phi, \mu\lambda).$$

Here, (ϕ, μ) signifies that ϕ is a Drinfeld μ-associator.

Proposition 13. *The action of* $\mathrm{GT}(\mathbb{C})$ *on* $\mathcal{S}$ *is free and transitive, making* $\mathcal{S}$ *a* $\mathrm{GT}(\mathbb{C})$*-torsor.*

Observe that the action of $\mathrm{GT}(k)$ on QUE algebras over k commutes with gauge transformations. In the latter case, the associativity and commutativity constraints are do not change, they are just given by a different sort of action. Indeed, gauge transformations lead to equivalent braided monoidal categories, while the kinds of transformations we dicuss in this section lead to inequivalent braided monoidal categories.

As a final remark, we mention that there is a canonnical homomorphism $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GT}(\mathbb{Q}_\ell)$ and there are close relations between pro-finite version of the Grothendieck-Teichmüller group is closely related to the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

References

[1] V. Chari and A. N. Pressley. *A Guide to Quantum Groups.* Cambridge University Press, 1995.

[2] V. G. Drinfeld. Quasi-hopf algebras. *Algebra i Analiz*, 1(6):114–148, 1989.

[3] V. G. Drinfeld. On quasitriangular quasi-hopf algebras and on a group that is closely connected with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. *Algebra i Analiz*, 2(4):149–181, 1990.

[4] C. Kassel. *Quantum Groups.* Springer-Verlag, 1995.