A glimpse into geometric representation theory

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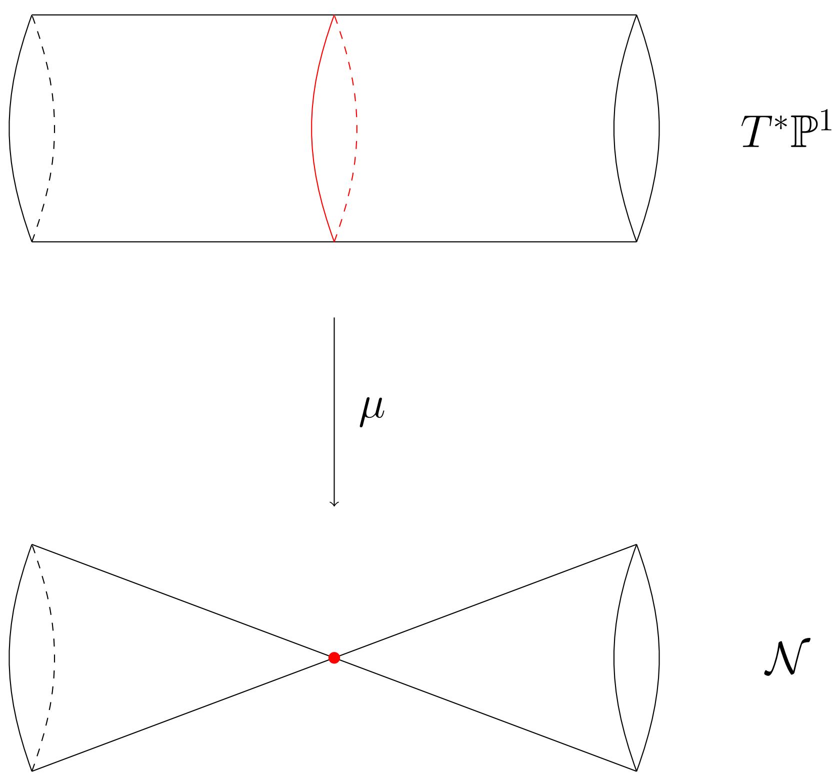
Abstract

We describe, in as elementary terms as possible, a bridge from linear algebra and representation theory on the one hand, to differential and algebraic geometry on the other. This bridge is known as the Beilinson–Bernstein localization theorem, and lies at the core of geometric representation theory. We will begin with defining polynomials and differential operators, and illustrate the key ideas behind the theorem via easy examples and calculations.



1 Introduction

This talk will be about a bridge between two areas of mathematics. On the one hand, we have linear algebra, abstract algebra, and representation theory, where there are many problems that are relatively easy to state but difficult to solve. These problems often come from other areas of mathematics, such as number theory, group theory, theoretical physics, etc. On the other hand we have differential and algebraic geometry, where there are also many hard problems inspired from other fields, but where we have powerful geometric techniques. There are many bridges between these two vast areas, and the actual picture is more complicated than this caricature, but it will work for our expository purposes. The bridge we'll focus on is known as the Beilinson–Bernstein localization theorem, and is one of the most important results of geometric representation theory.

2 Basics

Let $\mathbb{C}[x]$ be the set of polynomials in one variable x over the complex numbers. For example we have x^2, $(3+2i)x^3+7$, as well as constant polynomials such at 4. We can add and multiply polynomials, and these operations give $\mathbb{C}[x]$ the structure of an algebra. Multiplication distributes over addition, and we can subtract polynomials, but we cannot always divide.

We will also need the algebra $\mathbb{C}[x,y]$ of polynomials in two variables. Note that x and y commute. So that $x^3yx^7y^2$ is the same as x^10y^3.

We think of $\mathbb{C}[x]$ as the set of algebraic functions on one-dimensional complex space $\mathbb{C}^1$, and of $\mathbb{C}[x,y]$ as the set of algebraic functions on two-dimensional complex space $\mathbb{C}^2$. Part of the philosophy of algebraic geometry is that thinking about certain nice spaces (in this case $\mathbb{C}^1$ or $\mathbb{C}^2$) is essentially the same as studying its algebra of functions.

Definition 1. The operator of differentiation is given by taking derivatives of polynomials:

$$\partial : \mathbb{C}[x] \to \mathbb{C}[x]$$

$$f(x) \mapsto f'(x)$$

The operator of multiplication by x is given by:

$$x : \mathbb{C}[x] \to \mathbb{C}[x]$$

$$f(x) \mapsto xf(x)$$

We abuse notation slightly by using the same symbol x for an element of $\mathbb{C}[x]$ and an operator on $\mathbb{C}[x]$.

Proposition 2. *The operators ∂ and x satisfy the equation:*

$$\partial \circ x - x \circ \partial = \text{identity}.$$

Proof. Let $f(x) \in \mathbb{C}[x]$ be a polynomial. Then

$$(\partial \circ x - x \circ \partial)(f(x)) = \partial(xf(x)) - xf'(x) = f(x) + xf'(x) - xf'(x) = f(x).$$

□

Definition 3. The algebra of differential operators on $\mathbb{C}$ is defined as:

$$D_{\mathbb{C}} = \mathbb{C}\langle x, \partial \rangle / (\partial x - x\partial = 1).$$

This definition may look daunting, but it boils down to saying that we can think of differential operators sort of like polynomials in x and ∂, but now these two variables no longer commute. When we multiply, we can bring all x's to the left and all ∂'s to the left using the commutation rule. Example.

Notation: The commutator of two elements of an algebra is denoted $[a, b] = ab - ba$.

Definition 4. The algebra of differential operators on $\mathbb{C}^2$ is defined by

$$D_{\mathbb{C}^2} = \mathbb{C}\langle x, y, \partial_x, \partial_y \rangle / ([x, y] = [x, \partial_y] = [\partial_x, \partial_y] = [y, \partial_x] = 0, [\partial_x, x] = 1 = [\partial_y, y])$$

We define a few special elements in the algebra $D_{\mathbb{C}^2}$ as follows:

$$e = x\partial_y \qquad f = y\partial_x \qquad h = x\partial_x - y\partial_y$$

Exercise 5. Show that the following identities hold:

$$[h, e] = 2e \qquad [h, f] = -2f \qquad [e, f] = h.$$

3 Some geometry

We need to introduce one important object from geometry, namely projective space $\mathbb{P}^1$. We will abbreviate the set $\mathbb{C} \setminus \{0\}$ of nonzero complex numbers by $\mathbb{C}^\times$.

Definition 6. Let $\mathbb{P}^1$ be the quotient of $\mathbb{C}^2 \setminus \{(0, 0)\}$ by the equivalence relation $(a, b) \simeq (\lambda a, \lambda b)$ for $\lambda \in \mathbb{C}^\times$.

The space $\mathbb{P}^1$ is know as one-dimensional projective space over the complex numbers, and it parametrizes lines in $\mathbb{C}^2$. Topologically it is a sphere, so 2-dimensional over the real numbers and 1-dimensional over the complex numbers.

We were talking about polynomial functions on $\mathbb{C}^1$ and $\mathbb{C}^2$, and how they essentially tell you everything about the spaces. This is not so much the case for $\mathbb{P}^1$. Locally it is the same as $\mathbb{C}^1$, but globally it is quite different, which makes things in some sense more complicated and in another sense easier. In fact, there aren't any global polynomial functions on $\mathbb{P}^1$, but there are local ones.

In any case, one can make sens of differential operators $D_{\mathbb{P}^1}$. The operators $e, f, h \in D_{\mathbb{C}^2}$ descend (in a certain precise sense) to operators $\overline{e}, \overline{f}, \overline{h}$ in $D_{\mathbb{P}^1}$.

4 Some linear algebra

And now for something completely different (which will actually turn out to be not different at all).

Definition 7. The Lie algebra $\mathfrak{sl}_2$ is defined as as the set of two by two matrices with trace zero:

$$\mathfrak{sl}_2 = \{A \in \{\text{two by two matrices over } \mathbb{C}\} \ : \ \text{trace}(A) = 0\}$$

So if we write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with a, b, c, and d complex numbers, then the condition that the trace is zero becomes the equation $a + d = 0$. This might seem like a somewhat arbitrary subset of matrices, but the important property is the following:

Exercise 8. If A and B belong to $\mathfrak{sl}_2$ then so does their commutator $[A, B] = AB - BA$.

Also, it is slightly smaller than all matrices (three-dimensional over $\mathbb{C}$ rather than four dimensional). In fact, it is easy to see that $\mathfrak{sl}_2$ is spanned by the matrices $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So any matrix in $\mathfrak{sl}_2$ can be written as a linear combination of these matrices (with coefficients in $\mathbb{C}$).

Exercise 9. Prove the commutation relations

$$[H, E] = 2E \qquad [H, F] = -2F \qquad [E, F] = H$$

5 Easy case of the Beilinson–Bernstein Theorem

Combining our observations, we see that there is a function

$$\phi : \mathfrak{sl}_2 \to D_{\mathbb{C}^2}$$

determined by the assignments:

$$E \mapsto e \qquad F \mapsto f \qquad H \mapsto h$$

and extending linearly. Moreover, this function respects the brackets in the sense that

$$\phi([A, B]) = [\phi(A), \phi(B)]$$

for any two traceless matrices $A, B \in \mathfrak{sl}_2$. We now state the Beilinson–Bernstein theorem for $\mathfrak{sl}_2$.

Theorem 10. *The function ϕ extends to a surjective map of associative algebras*

$$\mu : U(\mathfrak{sl}_2) \to D_{\mathbb{P}^1}$$

whose kernel is generated by the kernel of the trivial character of the center of $U(\mathfrak{sl}_2)$. Moreover, there is an equivalence of categories between the category of representations of $\mathfrak{sl}_2$ with trivial central character and D-modules on $\mathbb{P}^1$.