Real and Quaternionic Representations of Finite Groups

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Abstract

Given a representation of a finite group G on a complex vector space V, one may ask whether it is the complexficiation of a representation of G on a real vector space. Another question is whether the representation is quaternionic, i.e. whether V has the structure of a quaternion module compatible with the action of G. The answers to these structural questions are closely related to the possible existence of a nondegenerate bilinear form on V fixed by G. We describe this relationship as well as the Schur indicator for distinguishing between various types of representations.

1 Definitions

If V is a vector space over a field F, we write $\mathrm{GL}(V)$ or $\mathrm{GL}(V, F)$ for the group of invertible F-linear transformations of V. We reserve the term "representation" for a linear action of a finite group G on a *complex* vector space V, that is, a homomorphism $G \to \mathrm{GL}(V, \mathbb{C})$.

Let V_0 be a real vector space, G a finite group, and $\rho : G \to \mathrm{GL}(V_0, \mathbb{R})$ a homomorphism. Consider the complex vector space V defined as

$$V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$$

obtained by extending scalars from $\mathbb{R}$ to $\mathbb{C}$. The linear action of G on V_0 naturally extends to a linear action on V given by

$$g \cdot (v \otimes \lambda) = (g \cdot v) \otimes \lambda.$$

So we obtain a representation $G \to \mathrm{GL}(V, \mathbb{C})$.

Definition. Let V be a complex vector space and G a finite group. A representation of G on V is *real* if there exists a real vector space V_0 and a homomorphism $G \to \mathrm{GL}(V_0, \mathbb{R})$ such that the representation $V_0 \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to V.

Consequently, an n-dimensional representation V is a real representation if and only if there is an n-dimensional real subspace V_0 of V that is invariant under the action of G. Since tensoring with $\mathbb{C}$ over $\mathbb{R}$ is just extending scalars, the character of any real representation is real. This can be seen explicitly by choosing a basis for V_0 and writing elements of G as matrices.

Example. Let $G = \langle x \mid x^4 \rangle$ be a cyclic group of order 4, and consider the homomorphism $G \to \mathrm{GL}_2(\mathbb{R})$ given by

$$x \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The resulting real representation of G on $\mathbb{C}^2$ is given by the same map, but now we think of the matrices as elements of $\mathrm{GL}_2(\mathbb{C})$ instead of $\mathrm{GL}_2(\mathbb{R})$. Under a change of basis, the representation on $\mathbb{C}^2$ can be written as

$$x \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Example. Let G be a nonabelian subgroup of $\mathrm{SU}(2)$. For instance, identify $\mathrm{SU}(2)$ withe the unit sphere in the quaternions and let

$$G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

We argue that the inclusion $G \hookrightarrow \mathrm{SU}(2) \subset \mathrm{GL}_2(\mathbb{C})$ is not a real representation. Toward a contradiction, suppose there is a 2-dimensional real subspace V_0 of $\mathbb{C}^2$ invariant under the action of G. Fix a basis $\mathcal{B}$ for V_0 as a real vector space, so each $g \in G$ acts as a matrix in this basis. Since V_0 is invariant under the action of each $g \in G$, the entries of the matrix g are real.

On the other hand, $\mathcal{B}$ is a basis for $\mathbb{C}^2$ as a complex vector space, and in this basis g is a matrix in $\mathrm{SU}(2)$. The real matrices in $\mathrm{SU}(2)$ form a copy of $\mathrm{SO}(2)$; it follows that G is a subgroup of $\mathrm{SO}(2) \simeq S^1$. This is a contradiction since S^1 is abelian and G is nonabelian.

Definition. A *quaternionic representation* of a finite group G is a complex representation V of G with a G-invariant conjugate linear endomorphism ψ such that $\psi^2 = -\mathrm{Id}$.

Thus, a quaternionic representation V of G has the structure of a module over the quaternions $\mathbb{H}$ where $i \in \mathbb{H}$ acts as left multiplication by $i \in \mathbb{C}$, $j \in \mathbb{H}$ acts by ψ, and $k \in \mathbb{H}$ acts by the composition of j and i. Moreover, the action of the quaternions on V commutes with the action of G on G.

Example. The representation of Q_8 on $\mathbb{C}^2$ in the previous example is a quaternionic representation with ψ defined by setting

$$\psi(1,0) = (0,-1) \qquad \psi(0,1) = (1,0)$$

and extending to all of $\mathbb{C}^2$ using conjugate linearlity.

Example. Consider the 1-dimensional representation of the cyclic group $G = \langle x \mid x^3 \rangle$ given by

$$\begin{gathered} \rho : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times \\ k \mapsto \omega^k \end{gathered}$$

where ω is a primitive third root of unity. Since ω is not real, multiplication by ρ does not preserve any 1-dimensional real subspaces. Hence ρ is not a real representation. To see that ρ is not a quaternionic representation, suppose ψ is a conjugate-linear endomorphism of $\mathbb{C}$. Then, for any $x \in \mathbb{C}$, we have $\psi(\omega x) = \bar{\omega}\psi(x)$, so ψ is not invariant under $\mathbb{Z}/3\mathbb{Z}$.

2 Bilinear forms fixed by G

In this section we show that real and quaternionic representations always have a nondegenerate bilinear form fixed by G. For real representations, the form is symmetric; it is skew-symmetric for quaternionic representations.

By convention, Hermitian forms are conjugate linear in the first coordinate and linear in the second coordinate. The Hermitian form H is *positive definite* if $H(v, v) > 0$ for nonzero v. In particular, positive definite Hermitian forms are nondegenerate. For the sake of conciseness, a "Hermitian form" will henceforth mean a positive definite Hermitian form.

Every representation V of G has a Hermitian form that is fixed by G. To see this, begin with any Hermitian form H_0 on V. For instance, choose a basis for V and identify H with the identity matrix. Define a form H_0 by

$$H(v, w) = \sum_{g \in G} H_0(gv, gw).$$

It is straighforward to check that H satisfies the desired properties.

A key point is that there is not always a nondegenerate bilinear form that is fixed by G. The averaging argument above does not preserve nondegeneracy since the sum of a collection of nonzero complex numbers could be zero. However, such a form exists for real representations, as the following proposition demonstrates.

Proposition. *If V is a real representation, then there is a nondegenerate symmetric bilinear form on V preserved by G. If V is irreducible, then this form is unique up to scaling.*

Proof. Say $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$. Start with any symmetric positive definite bilinear form B_0 on V_0. For example, choose a basis and identify B_0 with the identity matrix. Average over G to get a new positive definite symmetric bilinear form B on V_0 that is preserved by G:

$$B(v, w) = \frac{1}{|G|} \sum_{g \in G} B_0(gv, gw).$$

Now extend B to $V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$ by

$$B(v \otimes \lambda, w \otimes \mu) = \lambda \mu B(v, w).$$

A short computation verifies that B has the desired properties.

Suppose $\tilde{B}$ is another nondegenerate G-invariant symmetric bilinear form on V. Then B and $\tilde{B}$ define G-linear isomorphisms ϕ and $\tilde{\phi}$ between V and V^*. The composition

$$V \xrightarrow{\phi} V^* \xrightarrow{\tilde{\phi}^{-1}} V$$

is a G-linear. Schur's lemma now implies that B and $\tilde{B}$ differ by a scalar. $\square$

Proposition. *If V is an quaternionic representation, then there is a nondegenerate skew-symmetric bilinear form on V preserved by G. If V is irreducible, then this form is unique up to scaling.*

Proof. The last statement follows from arguments similar those in the previous proof, which apply to any nondegenerate bilinear form on a irreducible representation.

Let ψ be the conjugate linear G-invariant endomorphism of V with $\psi^2 = -1$. Let H be a Hermitian form on V fixed by G. Define

$$B(v, w) = H(\psi(v), w).$$

Then B is a bilinear form since ψ is conjugate linear and H is conjugate linear in the first factor.

To see that B is nondegenerate, suppose $B(v, w) = 0$ for all $w \in V$. Then $H(\psi(v), w) = 0$ for all $w \in V$. The nondegeneracy of H implies that $\psi(v) = 0$. Since ψ is invertible (its inverse is ψ^3) we conclude that $v = 0$. Thus B is nondegenerate. Similarly, the bilinear form $B'(v, w)$ defined as

$$B'(v, w) = B(v, w)$$

is nondegenerate.

Assume first that V is irreducible. Then Schur's lemma implies that B and B' differ by a scalar. Explicitly, there is a scalar $\lambda \in \mathbb{C}$ such that

$$B(v, w) = \lambda B'(v, w) = \lambda B(w, v)$$

for all $v, w \in V$. We argue the $\lambda = -1$, which will imply that B is skew-symmetric.

From the definition of B, we have

$$H(\psi(v), w) = \lambda H(\psi(w), v) = \lambda^2 H(\psi(v), w)$$

for all $v, w \in V$. Thus $\lambda^2 = 1$. Observe that

$$H(\psi(v), \psi(v)) = \lambda H(\psi^2(v), v) = -\lambda H(v, v)$$

for any $v \in V$. Since H is positive definite, it follows that λ is real and negative. We must have $\lambda^2 = 1$, so the only possibility is $\lambda = -1$.

Now suppose V is not irreducible. The irreducible factors of V are quaternionic representations themselves since ψ commutes with the action of G. The discussion in the irreducible case implies that the restriction of B to an irreducible factor is skew-symmetric. The factors can be chosen to be orthogonal with respect to the Hermitian form H, and hence with respect to B. Therefore, B skew-symmetric on all of V. $\square$

We will soon prove that a representation with a nondegenerate bilinear form fixed by G must be real or quaternionic. In particular, the converse of the first statements of both propositions in this section is true.

3 Bilinear forms for real-valued characters

As discussed earlier, necessary condition for a representation V to be real is that its character take on real values, i.e. $\chi_V(g) \in \mathbb{R}$ for all $g \in G$. This condition is not sufficient, as the following example shows.

Example. The group SU(2) consists of matrices of the form

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. Hence the trace of any element of SU(2) is real. The representation of $G = Q_8$ considered in a previous example therefore has real-valued character, even though it is a not real representation.

We will prove in this section that a representation with real-valued character is either real or quaternionic. The first step is to show that the possession of real-valued characters is equivalent to the existence of a nondegenerate bilinear form fixed by G. Then we will argue that this bilinear form is either symmetric, in which case the representation is real, or skew-symmetric, in which case the representation is quaternionic.

Proposition. *A representation V has real-valued characters if and only if there is a nondegenerate bilinear form on V fixed by G.*

Proof. In general, the characters of V and V^* are related by conjugation. So the character of V is real-valued if and only if V and V^* have the same characters. Since a representation is determined by its character, the character of V is real-valued if and only if V and V^* are isomorphic representations of G.

Suppose $\phi : V \to V^*$ be a G-linear isomorphism. Define a bilinear form on V as $B(v, w) = \phi(v)(w)$. An quick computation verifies that B is nondegenerate and fixed by G.

Conversely, if there is a nondegenerate bilinear form B fixed by G, then it defines a G-module isomorphism between V and V^* by $v \mapsto B(v, -)$. □

Assume that the character of V is real-valued and $\phi : V \to V^*$ is a G-linear isomorphism as in the preceding proof. Another nondegenerate bilinear form on V fixed by G is given by $B'(v, w) = \phi(w)(v) = B(w, v)$. Since V is irreducible, these forms are related by a scalar, say ϵ. Thus,

$$B(v, w) = \epsilon B'(v, w) = \epsilon^2 B(v, w),$$

so $\epsilon^2 = 1$. In other words, B is either symmetric ($\epsilon = 1$) or skew symmetric ($\epsilon = -1$).

Let H be a Hermitian form on V fixed by G, so H defines an G-linear isomorphism $H \colon V \to V^*$. Define a map ψ on V as the composition

$$\psi : V \xrightarrow{\phi} V^* \xrightarrow{H^{-1}} V.$$

Note that ψ is conjugate linear and commutes with the action of G. Therefore $\psi^2 : V \to V$ is G-linear. By Schur's Lemma, $\psi^2 = \lambda \mathrm{Id}$.

Observe that

$$H(\psi(v), w) = B(v, w) = \epsilon B'(v, w) = \epsilon B(w, v) = \epsilon H(\psi(w), v)$$

for any $v, w \in V$. So in particular we have

$$H(\psi(v), \psi(v)) = \epsilon H(\psi^2(v), v) = \epsilon\lambda H(v, v).$$

Since H is positive definite, the product $\epsilon\lambda$ must be real and positive. In particular, λ is real with sign equal to ϵ. Without any loss, we may replace H by $\sqrt{|\lambda|}H$ in order to obtain, after a straightforward computation, that $\psi^2 = \epsilon\mathrm{Id}$. We have two cases:

- Suppose first that $\epsilon = 1$. Then $\psi^2 = \mathrm{Id}$ and ψ is conjugate linear; in particular, it is $\mathbb{R}$-linear. View V as a real vector space (of double the dimension), so ψ is a $\mathbb{R}$-linear transformation of V of order 2. Thus, ψ is diagonalizable with eigenvalues ± 1. Decompose V into eigenspaces for the action of ψ as

$$V = V_+ \oplus V_-$$

where ψ acts on the first factor as the identity and on the second as multiplication by -1. Since ψ is conjugate linear, it is easy to show that $iV_+ = V_-$ and $iV_- = V_+$. So the (real) dimensions of V_+ and V_- are equal to each other, and hence to the (complex) dimension of V. Moreover, V_+ is G-invariant: if $v \in V_+$ and $g \in G$, then

$$\psi(gv) = g\psi(v) = gv,$$

since the action of G commutes with ψ. Therefore, V is real representation.

- Now suppose that $\epsilon = -1$. Then, equipped with ψ, V becomes a quaternionic representation of G.

The table below summarizes the results so far. By a slightly confusing convention, a representation whose character takes non-real values is called *complex.*

Type of representation	character values	nondegenerate bilinear form fixed by G?
Real	real	Yes, symmetric
Quaternionic	real	Yes, skew-symmetric
Complex	not all real	No

4 The Schur indicator

We begin with the definition of the Schur indicator. We will see later in the section how the Schur indicator distinguishes between the types of representations given in the table above.

Definition. Let V be a representation of G with character χ. The *Schur indicator* for V is defined as

$$\sigma(V) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Bilinear forms are elements of the tensor product $V^* \otimes V^*$. If V is a representation G, then $V^* \otimes V^*$ inherits an action of G in a standard way (the precise description is omitted here). There is a canonical decomposition of $V^* \otimes V^*$ into G-invariant subspaces:

$$V^* \otimes V^* \simeq \mathrm{Sym}^2 V^* \oplus \bigwedge^2 V^*.$$

That is, any bilinear form on V is the sum of a symmetric and a skew-symmetric bilinear form. The space of forms fixed by G is

$$(V^* \otimes V^*)^G \simeq (\mathrm{Sym}^2 V^*)^G \oplus (\bigwedge^2 V^*)^G.$$

If χ is the character of V and χ_{triv} is the character of the trivial representation, then, using the formulas for the characters of the symmetric and exterior powers, we obtain

$$\begin{aligned}\dim(\mathrm{Sym}^2 V^*)^G = \langle \chi_{\mathrm{triv}}, \chi_{\mathrm{Sym}^2 V^*} \rangle &= \frac{1}{2|G|} \sum_{g \in G} (\chi(g)^2 + \chi(g^2)) \\ &= \frac{1}{2|G|} \sum_{g \in G} \chi(g)^2 + \frac{\sigma(V)}{2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\dim(\bigwedge^2 V^*)^G = \langle \chi_{\mathrm{triv}}, \chi_{\bigwedge^2 V^*} \rangle &= \frac{1}{2|G|} \sum_{g \in G} (\chi(g)^2 - \chi(g^2)) \\ &= \frac{1}{2|G|} \sum_{g \in G} \chi(g)^2 - \frac{\sigma(V)}{2}.\end{aligned}$$

Consequently,

$$\dim(V^* \otimes V^*)^G = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2.$$

Assume that V is irreducible. If the character χ of V is real-valued, then there is a nondegenerate bilinear form fixed by G, and it is unique up to scaling. Thus

$$1 = \dim(V^* \otimes V^*)^G = \frac{1}{|G|} \sum_{g \in G} \chi(g)^2.$$

This form is symmetric if and only if V is real, in which case we must have

$$1 = \dim(\mathrm{Sym}^2 V^*)^G = \frac{1}{2} + \frac{\sigma(V)}{2},$$

so $\sigma(V) = 1$. The form is skew-symmetric if and only if V is quaternionic, in which case

$$1 = \dim(\bigwedge^2 V^*)^G = \frac{1}{2} - \frac{\sigma(V)}{2},$$

so $\sigma(V) = -1$. The last step is to show that complex represntations (i.e. representations whose characters take on non-real values) have Schur indicator 0. For this we require the following consequence of Schur's lemma.

Lemma. *Any nonzero G-invariant bilinear form on a irreducible representation of G is nondegenerate.*

Proof. Let B be nonzero G-invariant bilinear from on V, so B defines a non-zero G-linear homomorphism $\phi : V \to V^*$. If V is irreducible, then so is V^*. By Schur's lemma, ϕ is an isomorphism. Hence B is nondegenerate. $\square$

Now suppose that V is a complex representation. Then, by previous results, there is no nondegenerate bilinear G-invariant form on V. The lemma implies that $(V^* \otimes V^*)^G = 0$. Hence

$$\frac{1}{|G|}\sum_{g \in G} \chi(g)^2 = 0 \qquad \text{and} \qquad (\mathrm{Sym}^2 V^*)^G = 0,$$

from which it follows that $\sigma(V) = 0$.

5 Overview of the main results

Each irreducible representation V of G lends itself to one of the descriptions below.

- The character of V is real-valued and there is a symmetric nondegenerate bilinear form on V preserved by G. In this case V is a real representation and $\sigma(V) = 1$.
- The character of V is real-valued and there is a skew-symmetric nondegenerate bilinear form on V preserved by G. In this case V is a quaternionic representation and $\sigma(V) = -1$.
- The character of V takes on non-real values. In this case V is a complex representation, there is no nondegenerate bilinear form on V preserved by G, and $\sigma(V) = 0$.

The types of representations and their key properties are summarized in the table below.

Type of representation	character values	nondegenerate bilinear form fixed by G?	Schur indicator
Real	real	Yes, symmetric	1
Quaternionic	real	Yes, skew-symmetric	-1
Complex	not all real	No	0