Notes on the Radon transform

Iordan Ganev

This document stems out of the author's notes from reading the paper 'Characteristic varieties of character sheaves' by I. Mirković and K. Vilonen [VM88], and from discussions with Quoc Ho and Anton Mellit. The author claims no originality.

1 Preliminaries

Let G be a connected reductive group, B a Borel subgroup, and T a maximal torus contained in B. Let N be the unipotent radical of B, so that $B = N \rtimes T$. Let $W = N_G(T)/T$ be the Weyl group and $n = \dim(G/B)$. We denote by $D(X)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves on an algebraic variety X. For a morphism $f : X \to Y$, we have the usual functor f_*, $f_!$, f^*, and $f^!$ between $D(X)$ and $D(Y)$. Recall that if f is proper then $f_* = f_!$ and if f is smooth with fibers of dimension d then $f^! = f^*[2d]$. In the latter case, we set $f^0 = f^![d]$, which is an exact functor with respect to the perverse t-structure, and hence preserves perverse sheaves.

2 The Radon transform

We follow Section 3 of [VM88]. There is a right action of T on G/N is given by $(xN)\cdot t = xtN$. This is well-defined since T normalizes N. We consider the diagonal right action of T on $G/N \times G/N$

Definition 2.1. We denote by Y the quotient of $G/N \times G/N$ by the right diagonal action of T.

Lemma 2.2. *There is a well-defined map*

$$r : G \times G/B \to Y = (G/N \times G/N)/T$$

given by $(g, xB) \mapsto [gxB, xB]$. *The map* r *is a fibration with fibers isomorphic to* N.

Proof. To see that this map is well-defined, suppose $xB = yB$. Then $y = xb$ for some $b = tn = (t, n) \in B = T \ltimes N$. It follows that $(gyN, yN) = (gxtnN, gxtnN) = (gxtN, gxtN)$, which is equivalent under the diagonal right T-action to (gxN, xN). Finally, one computes that the fiber of r over $[xN, yN]$ to be $xNy^{-1} \times \{yN\}$. □

Consider the diagram:

$$G \xleftarrow{\;q\;} G \times G/B \xrightarrow{\;r\;} Y \tag{2.1}$$

where $q(g, xB) = g$ is the projection map.

Definition 2.3. The Radon transform is the functor $R = r_* q^* : D(G) \to D(Y)$. We also define a functor $\check{R} = q_* r^* : D(Y) \to D(G)$.

We note that both maps are smooth with fibers of dimension n, and that q is proper, we have that $r^! = r^*[2n]$, $q^! = q^*[2n]$, $q_* = q_!$, and $\check{R}[2n] = q^! r_!$ is left adjoint to R.

Definition 2.4. The category $D(G)$ carries a convolution tensor structure defined as $- \star - = (p_2)_*(p_1^*(-) \otimes m^*(-))$ where $m : G \times G \to G$ is given by $m(g,h) = g^{-1}h$ and $p_1, p_2 : G \times G \to G$ are the projection maps.

Remark 2.5. One can replace m^* by $m^0 = m^![-\dim G]$ in order to guarantee that the convolution of perverse sheaves is perverse.

Definition 2.6. Let $\mathcal{N} \subseteq G$ be the unipotent cone of G, and let

$$\sigma : \tilde{\mathcal{N}} = \{(g, xB) \in G \times G/B \mid x^{-1}gx \in N\} \to G$$

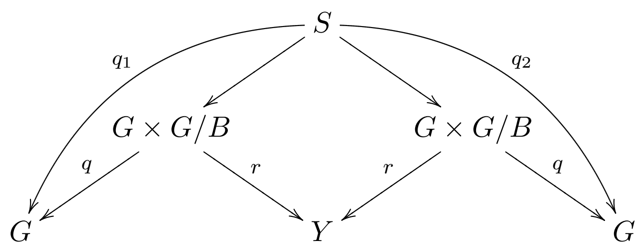
be the Springer map, which is a resolution of the unipotent cone $\mathcal{N}$, defined by $\sigma(g, xB) = g$. Let $\mathrm{Spr} = \sigma_* \overline{\mathbb{Q}}_{\ell, \tilde{\mathcal{N}}}$ be the Springer sheaf.

Proposition 2.7. *We have that*

$$\check{R} \circ R = \mathrm{Spr} \star -$$

as endofunctors of $D(G)$.

Proof. To compute the composition $\check{R} \circ R$, consider the following diagram:



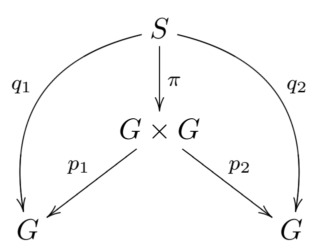
where S is the fiber product $(G \times G/B) \times_Y (G \times G/B)$. Explicitly,

$$S = \{(g, h, xB, yB) \in G \times G \times G/B \times G/B \ : \ [gxN, xN] = [hyN, yN] \in (G/N \times G/N)/T\}$$

Thus, if $(g, h, xB, yB) \in S$, then there exists $t \in T$ such that $gxN = hytN$ and $xN = ytN$. These equations imply that $gxN = hxN$ anbd $xB = yB$, and hence we obtain the following equivalent definition of S:

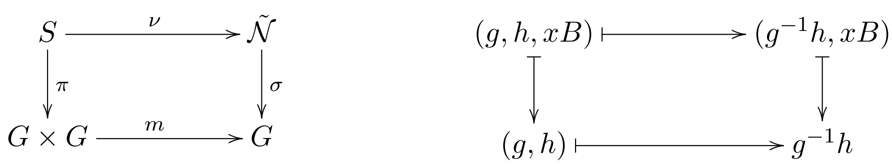
$$S = \{(g, h, xB) \in G \times G \times G/B \ : \ gxN = hxN \in G/N\}$$

We will also use the diagram:



(2.2)

where $\pi = q_1 \times q_2$. The total space $\tilde{\mathcal{N}}$ of the Springer resolution can be identified with the preimage of $1 \times G$ under the map π, and that π restricts to σ along this preimage. It is straightforward to verify that the following diagram is Cartesian:



(2.3)

Finally, we compute:

$$\begin{aligned}\check{R}R(\mathcal{F}) &= (q_*r^*r_*q^*)(\mathcal{F}) = ((q_2)_*q_1^*)(\mathcal{F}) = ((p_2)_*\pi_*\pi^*p_1^*)(\mathcal{F}) = (p_2)_*\left[p_1^*\mathcal{F}\otimes\pi_*\overline{\mathbb{Q}}_{\ell,S}\right]\\ &= (p_2)_*\left[p_1^*\mathcal{F}\otimes\pi_*\nu^*\overline{\mathbb{Q}}_{\ell,\tilde{\mathcal{N}}}\right] = (p_2)_*\left[p_1^*\mathcal{F}\otimes m^*\sigma_*\overline{\mathbb{Q}}_{\ell,\tilde{\mathcal{N}}}\right] = (p_2)_*\left[p_1^*\mathcal{F}\otimes m^*\mathrm{Spr}\right] = \mathcal{F}\star\mathrm{Spr}\end{aligned}$$

where we base change in the second equality, Diagram 2.2 in the third, the projection formula in the fourth, and base change along Diagram 2.3 in the sixth. □

Remark 2.8. Note that S and $G\times G$ each carry a G action by left translations in the first two factors and ν is equivariant. Furthermore, S and G, and can be identified (using the maps ν and m) with $G\times\tilde{\mathcal{N}}$ and $G\times G$, respectively, where G acts freely on the first factor. Under these identifications, ν becomes $1\times\sigma$.

3 The stacky version of the Radon transform

The group G acts on each of the spaces in Diagram 2.1. Explicitly, G acts on G by conjugation, on $G\times G/B$ by conjugation in the first factor and left translation in the second, and on Y by left translations in both factors. Quotienting out by the action of G, we obtain the stacky version of Diagram 2.1:

$$\begin{array}{ccc} & \frac{G}{B} & \\ {}^{q}\swarrow & & \searrow^{r} \\ \frac{G}{G} & & N\backslash\frac{G}{T}/N \end{array} \tag{3.1}$$

The convolution on G can be defined using the stacky diagram:

$$\begin{array}{ccc} & \frac{G\times G}{G} & \xrightarrow{p_2} \frac{G}{G} \\ {}^{p_1}\swarrow & & \searrow^{m} \\ \frac{G}{G} & & \frac{G}{G} \end{array}$$

where $\frac{G\times G}{G}$ is the quotient of $G\times G$ by the simultaneous conjugation action of G. Let $G\times_B N$ denote the balanced product of G and N over B, where B acts by right translations on G and by conjugation on N. There is an isomorphism

$$\tilde{\mathcal{N}}\to G\times_B N$$

$$(g,xB)\mapsto[x,x^{-1}gx].$$

Under this isomorphism, the action of G on $\tilde{\mathcal{N}}$ is identified with the left translation action of G on $G\times_B N$, and taking the quotient, we obtain

$$G\backslash\tilde{\mathcal{N}} = G\backslash(G\times_B N) = \frac{N}{B},$$

that is, the quotient of N by the conjugation action of B. Thus, the stacky version of the Springer resolution is the map

$$\sigma:\frac{N}{B}\longrightarrow\frac{G}{G}$$

induced by the inclusion of N into G. Finally, let $\frac{G\times N}{B}$ denote the quotient of $G\times N$ by the simultaneous conjugation action of B.

Lemma 3.1. *We have:*

1. *There is a Cartesian diagram:*

$$\begin{array}{ccc} \frac{G\times N}{B} & \longrightarrow & \frac{G}{B} \\ \downarrow & & \downarrow{\scriptstyle r} \\ \frac{G}{B} & \xrightarrow{\ r\ } & N\backslash\frac{G}{T}/N \end{array}$$

   In other words, the fiber product of $\frac{G}{B}$ *with itself over* $N\backslash\frac{G}{T}/N$ *can be identified with* $\frac{G\times N}{B}$*.*

2. *There is a Cartesian diagram:*

$$\begin{array}{ccc} \frac{G\times N}{B} & \longrightarrow & \frac{N}{B} \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ \frac{G\times G}{G} & \xrightarrow{\ m\ } & \frac{G}{G} \end{array}$$

   where the left map takes the equivalence class of (g, n) *to that of* $(g, n^{-1}g)$*.*

Sketch of proof. To see the first claim, suppose $[g], [h] \in \frac{G}{B}$ are B-conjugacy classes whose images in $N\backslash\frac{G}{T}/N$ coincide. Then there exist $n_1, n_2 \in N$ and $t \in T$ such that $h = n_1 t g t^{-1} n_2$. Let $b = t^{-1} n_1^{-1}$. Then $[h] = [bhb^{-1}] = [gt^{-1}n_1n_2t]$. Thus we obtain an element $n = t^{-1}n_1n_2t \in N$. The pair (g, n) is determined up to simultaneous B-conjugacy. To see the second claim, suppose $[g, h] \in \frac{G\times G}{G}$ and $[n] \in \frac{N}{B}$ are such that gh^{-1} and n are conjugate in G. Then $h = x^{-1}n^{-1}xg$ for some $x \in G$. Thus, the conjugacy class of h is determined by g and n, and the pair (g, n) is determined up to simultaneous B-conjugacy. $\square$

4 A different version of the Radon transform

This section is based on communication with A. Mellit. An equivalent version of Diagram 3.1 is given by:

$$\begin{array}{ccccc} & B\backslash(G\times B)/B & & & \\ \swarrow & & \searrow & & \\ G\backslash(G\times G)/G & & & B\backslash(G\times T)/B & \end{array} \qquad \begin{array}{ccc} & [g,b] & \\ \swarrow & & \searrow \\ [g,b] & & [g,\pi(b)] \end{array}$$

where in all cases we take the quotient by the diagonal left and right multiplication actions. In the case of $B \times B$ acting on T, we have $(b_1, b_2)t = \pi(b_1)t\pi(b_2^{-1})$, where $\pi : B \to T$ is the quotient map, and so the stabilizer of $1 \in T$ is given by the fiber product of B with itself over T, that is:

$$B \times_T B = \{(b_1, b_2) \mid \pi(b_1) = \pi(b_2)\} \simeq N \times T_\Delta \times N.$$

Let $G\backslash(G \times G) \times_G (G \times G)/G$ denote the balanced product $G\backslash(G \times G)$ with $(G \times G)/G$ under the internal action of G, that is, the quotient of $G \times G \times G \times G$ by the action of $G \times G \times G$ given by $(x, y, z)\cdot(g_1, g_2, h_1, h_2) = (xg_1y^{-1}, xg_2y^{-1}, yh_1z^{-1}, yh_2z^{-1})$. The orbits are identified with the orbits of G on $G \times G$ acting by simultaneous conjugation, via the map $(g_1, g_2, h_1, h_2) \mapsto (g_2^{-1}g_1, h_1h_2^{-1})$.

Lemma 4.1. *We have:*

1. *There is a Cartesian diagram:*

$$\begin{array}{ccc} B\backslash(G\times(B\times_T B))/B & \longrightarrow & B\backslash(G\times B)/B \\ \downarrow & & \downarrow r \\ B\backslash(G\times B)/B & \xrightarrow{\ r\ } & B\backslash(G\times T)/B \end{array}$$

   In other words, the fiber product of $B\backslash(G\times B)/B$ *with itself over* $B\backslash(G\times T)/B$ *can be identified with* $B\backslash(G\times(B\times_T B))/B$.

2. *There is a Cartesian diagram:*

$$\begin{array}{ccc} B\backslash(G\times(B\times_T B))/B & \longrightarrow & B\backslash(B\times_T B)/B \\ \downarrow & & \downarrow \\ G\backslash(G\times G)\times_G(G\times G)/G & \longrightarrow & G\backslash(G\times G)/G \end{array}$$

   where the left map is given by $[g,b_1,b_2]\mapsto[g,b_1,g,b_2]$*, the bottom map by* $[g_1,g_2,h_1,h_2]\mapsto[g_2^{-1}g_1h_2,h_1]$*, the top map is the projection onto the last two factors, and the right map is induced by the inclusion of* $B\times B$ *into* $G\times G$.

Sketch of proof. The first claim follows from the non-equivariant version: $(G\times B)\times_{G\times T}(G\times B)=G\times(B\times_T B)$. The second claim reduces to the proof of the second item in Lemma 3.1. □

Note that B acts by conjugation on $N=\ker(\pi)$. The total space of the Springer resolution is $\tilde{\mathcal{N}}=G\times_B N$, where we take the balanced product for the right multiplication action of B on G, and the conjugation action of B on N. The quotient of $\tilde{\mathcal{N}}$ by the left multiplication action of G is identified with the quotient $\frac{N}{B}$ of N by the conjugation action of B. We see the connection with Springer theory through the following lemma.

Lemma 4.2. *We have* $B\backslash(B\times_T B)/B=\frac{N}{B}$.

Proof. Indeed, one checks that the map

$$B\times_T B\to N,\qquad (b_1,b_2)\mapsto b_1b_2^{-1}$$

is well-defined and surjective, and the preimage of any B-orbit on N is a $B\times B$ orbit on $B\times_T B$. □

5 Local systems

In this section, we clarify a claim appearing in Section 2.1 of [VM88]. For each $w\in W$, we fix a lift $\dot{w}\in N_G(T)\subseteq G$. We define the w-twisted conjugation action of T on itself as:

$$a:T\times T\to T$$
$$s,t\mapsto \dot{w}s\dot{w}^{-1}ts^{-1},$$

which is well-defined since $\dot{w}\in N_G(T)$ and T is abelian. We will consider the w-twisted adjoint quotient $\frac{T}{T^w}$. We make sense of the following claim[1].

[1]The argument is based discussions with Q. Ho.

Lemma 5.1. *There is natural bijection between the set of (isomorphism classes of) rank one local systems on $\frac{T}{T^w}$ and the set of (isomorphism classes) of local systems on T that are w-invariant.*

Proof. We identify the fundamental group of T with the cocharacter lattice $X_*(T)$, and write $\alpha \mapsto w(\alpha)$ for the action of $w \in W$ on $\alpha \in X_*(T)$. A rank one local system on T amounts to a representation of the fundamental group $\pi_1(T) \simeq X_*(T)$ into $\mathbb{C}^\times$ (modulo the trivial conjugation action of $\mathbb{C}^\times$ on itself). Thus, we have an action of W on the set of rank one local systems on T. A rank one local system on the w-twisted adjoint quotient $\frac{T}{T^w}$ is rank one local system on T together with an isomorphism between the pullbacks under a and under the projection $\mathrm{pr}_2 : T \times T \to T$. In terms of fundamental groups, we have that a rank one local system on $\frac{T}{T^w}$ is representation $\rho : X_*(T) \to \mathbb{C}^\times$ together with an isomorphism between the pullbacks under the two maps

$$a_* : X_*(T) \times X_*(T) \to X_*(T) \qquad (\mathrm{pr}_2)_* : X_*(T) \times X_*(T) \to X_*(T)$$

$$(\alpha, \beta) \mapsto w(\alpha) + \beta - \alpha \qquad (\alpha, \beta) \mapsto \beta$$

This condition simplifies to $\rho(w(\alpha)) = \rho(\alpha)$ for all $\alpha \in X_*(T)$. □

6 Miscellaneous

Lemma 6.1. *Suppose G acts on a set X and H is a subgroup of G. Then the set of orbits of the diagonal action of G on $X \times G/H$ are in bijection with the orbits of H on X:*

$$G\backslash(X \times G/H) = X/H.$$

Proof. The map $\phi : X \to X \times G/H$ by $\phi(x) = (x, H)$ induces an isomorphism $X/H \to G\backslash(X \times G/H)$. The inverse map can be described as $(x, gH) \mapsto [xg]$. Note that we can identify the left action of G on a set X with a right action via $x \cdot g = g^{-1} \cdot x$. □

Lemma 6.2. *There is a G-equivariant bijection*

$$(G/N \times G/N)/T = G \times_B G/N.$$

Sketch of proof. The map $\phi : G \times G \to G \times_B G/N$ defined by $\phi(g, h) = [g, g^{-1}hN]$ induces the desired bijection. An inverse is induced by the map $\psi : G \times G/N \to (G/N \times G/N)/T$, where $\psi(x, yN) = (xN, xyN)T$. One verifies easily that (1) $\psi(gtn_1, gtn_2) = \psi(g, h)$ for any $t \in T$ and $n_1, n_2 \in N$, (2) $\phi(xb, b^{-1}ybN) = \phi(x, yN)$ for any $b \in B$, and (3) G-equivariance, that is, $\phi(\gamma \cdot (g, h)) = \gamma \cdot \phi(g, h)$ for any $\gamma \in G$. □

References

[VM88] K. Vilonen and I. Mirkovic, *Characteristic varieties of character sheaves.*, Inventiones mathematicae **93** (1988), no. 2, 405–418.