NOTES ON LUSZTIG'S NON-ABELIAN FOURIER TRANSFORM

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1. Preliminaries

Let G be a finite group. The group algebra of G is the vector space $\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$ of functions from G to $\mathbb{C}$, equipped with the convolution product:

$$\begin{aligned} \mathbb{C}[G] \otimes \mathbb{C}[G] &\to \mathbb{C}[G] \\ f \otimes g &\mapsto [x \mapsto \sum_{y \in G} f(xy^{-1})g(y)] \end{aligned}$$

We write $f * g$ for the convolution of $f, g \in \mathbb{C}[G]$. For $x \in G$, let $e_x \in \mathbb{C}[G]$ denote the characteristic function of x, so that $\{e_x\}_{x \in G}$ forms a basis of $\mathbb{C}[G]$, and any $f \in \mathbb{C}[G]$ can be expressed as $f = \sum_{x \in G} f(x) e_x$. Any representation $\rho : G \to \mathrm{GL}(V)$ of G extends to a representation of the group algebra $\mathbb{C}[G]$,

$$\tilde{\rho} : \mathbb{C}[G] \to \mathrm{End}(V),$$

by setting $\tilde{\rho}(e_x) = \rho(x)$. This assignment establishes an equivalence between the category of representations of G and the category of representations of the group algebra $\mathbb{C}[G]$. Let $Z_G = Z(\mathbb{C}[G])$ denote the center of the group algebra $\mathbb{C}[G]$. One computes that Z_G comprises the class functions, i.e., functions constant on conjugacy classes of G:

$$Z_G = \{f \in \mathbb{C}[G] \mid f(g) = f(xgx^{-1}) \text{ for all } g, x \in G\}.$$

We use the notation $\frac{G}{G}$ for the set of conjugacy classes of G, so that, as a vector spaces, Z_G is the space $\mathbb{C}\left[\frac{G}{G}\right]$ of functions on $\frac{G}{G}$. A source of class functions is the characters of finite-dimensional representations. Specifically, if $\rho : G \to \mathrm{GL}(V)$ is a finite-dimensional representation of G, the character

$$\chi_V : G \to \mathbb{C}$$

is defined as $\chi_V(g) = \mathrm{trace}(\rho(g))$, and is a class function. Now suppose ρ is an irreducible representation of G, and consider the corresponding representation $\tilde{\rho} : \mathbb{C}[G] \to \mathrm{End}(V)$ of the group algebra. Then the center Z_G acts by scalars on V; in other words, $\tilde{\rho}(z)$ is a scalar matrix for every $z \in Z_G$. The central character of ρ is defined by:

$$\lambda_\rho : Z_G \to \mathbb{C}; \qquad z \mapsto \frac{\mathrm{trace}(\tilde{\rho}(z))}{\dim V}.$$

We can express the central character λ_ρ in terms of the character χ_ρ:

$$\lambda_\rho(z) = \frac{1}{\chi_\rho(1)} \sum_{g \in G} z(g) \chi_\rho(g).$$

In what follows, we fix $\{\rho_i : G \to \mathrm{GL}(V_i)\}_{i=1}^r$ to be (representatives of the isomorphism classes of) the finite-dimensional irreducible representations of G. The characters $\{\chi_{V_i}\}$ define an (orthonormal) basis of the set of class functions.

2. Two-class functions

Consider the set $\{(x,y) \in G \times G \mid [x,y] = 1\}$ consisting of pairs of commuting elements of G. The group G acts on this space by simultaneous conjugation, and functions on the set of orbits are called two-class functions:

Definition 2.1. The space of two-class functions on G is defined as:

$$Z_G^{(2)} = \mathbb{C}\left[\{(x,y) \in G \times G \mid [x,y] = 1\}/G\right],$$

Observe that we have projections onto each coordinate:

$$\begin{array}{ccc} & \{(x,y) \in G \times G \mid [x,y] = 1\}/G & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \frac{G}{G} & & \frac{G}{G} \end{array}$$

The fiber of π_1 over the conjugacy class of $x \in G$ is identified with the set $\frac{C_G(x)}{C_G(x)}$ of conjugacy classes of the centralizer $C_G(x)$ of x in G. Similarly, the fiber of π_2 over the conjugacy class of $y \in G$ is identified with the set of conjugacy classes of the centralizer of y. Thus, we have two isomorphisms:

$$\phi_1 : \bigoplus_{g \in \frac{G}{G}} Z_{C_G(g)} \xrightarrow{\sim} Z_G^{(2)} \qquad \phi_2 : \bigoplus_{g \in \frac{G}{G}} Z_{C_G(g)} \xrightarrow{\sim} Z_G^{(2)},$$

between $Z_G^{(2)}$ and the direct sum of class functions on the centralizers $C_G(g)$ as g runs over the conjugacy classes of G.

Definition 2.2. Let $\mathcal{M}(G)$ be the quotient of the set

$$\{(g,\sigma) \mid g \in G,\ \sigma \text{ is an irreducible charater of the centralizer } C_G(g)\}$$

by the action of G given by $h \triangleright (g,\sigma) = (hgh^{-1}, z \mapsto \sigma(h^{-1}zh))$.

We observe that every element $m = (g,\sigma)$ of $\mathcal{M}(G)$ defines an element z_m of $\bigoplus_{g \in \frac{G}{G}} Z_{C_G(g)}$, and, moreover, the set $\{z_m\}_{m \in \mathcal{M}(G)}$ comprises a basis. Hence, we obtain two bases of the space of two-class functions: $\{\phi_1(z_m)\}$ and $\{\phi_2(z_m)\}$.

Lemma 2.3. *The change-of-basis matrix between the bases $\{\phi_1(z_m)\}$ and $\{\phi_2(z_m)\}$ of $Z_G^{(2)}$ is given by:*

$$\{\phi_1(z_m), \phi_2(z_{m'})\} = \frac{1}{|C_G(x)||C_G(x')|} \sum_{\substack{g \in G \\ gxg^{-1} \in C_G(x')}} \sigma(gx'g^{-1})\overline{\sigma'(g^{-1}xg)},$$

where $m = (x,\sigma)$ and $m' = (x',\sigma')$ are elements of $\mathcal{M}(G)$.

Proof. There is a non-degenerate inner product on $Z_G^{(2)}$ given by:

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{\substack{(w,u) \in G^2 \\ [w,u]=1}} \alpha(w,u)\overline{\beta(w,u)}$$

Each of the bases $\{\phi_1(z_m)\}$ and $\{\phi_2(z_m)\}$ are orthonormal for this inner product. Hence the entries of the change of basis matrix is given by:

$$\langle \phi_1(z_m), \phi_2(z_{m'}) \rangle = \frac{1}{|G|} \sum_{\substack{(w,u)\in G^2 \\ [w,u]=1}} \phi_1(z_m)(w,u)\overline{\phi_2(z_{m'})(w,u)}$$

Using the fact that $\phi_1(z_m)$ is supported on the fiber $\pi_1^{-1}(x)$, and the fact that it is invariant for the conjugation action of G, the above expression reduces to:

$$= \frac{1}{|C_G(x)|} \sum_{u\in C_G(x)} \phi_1(z_m)(x,u)\overline{\phi_2(z_{m'})(x,u)} = \frac{1}{|C_G(x)|} \sum_{u\in C_G(x)} \sigma(u)\overline{\phi_2(z_{m'})(x,u)}$$

Note that $\phi_2(z_{m'}(x,u)) = 0$ unless u is conjugate to x'. We obtain:

$$= \frac{1}{|C_G(x)||C_G(x')|} \sum_{\substack{g\in G \\ gx'g^{-1}\in C_G(x)}} \sigma(gx'g^{-1})\overline{\phi_2(z_{m'})(x,gx'g^{-1})}$$

Finally, $\phi_2(z_{m'})(x,gx'g^{-1}) = \phi_2(z_{m'})(g^{-1}xg,x') = \sigma'(g^{-1}xg)$, and the result follows. □

Remark. If G is abelian, then $\mathcal{M}(G) = G \times \hat{G}$, and Lusztig's non-abelian Fourier transform reduces to the usual Fourier transform (see the Appendix below).

3. G-equivariant vector bundles

Suppose G acts on a finite set X.

Definition 3.1. A G-equivariant vector bundle on X is the data of a finite-dimensional vector space V_x for every $x \in X$, together with an isomorphism:

$$\alpha_{g,x} : V_x \xrightarrow{\sim} V_{gx}$$

for any $g \in G$ and $x \in X$, that satisfy $\alpha_{h,gx} \circ \alpha_{g,x} = \alpha_{hg,x}$ and $\alpha_{1,x} = \mathrm{Id}$. We denote by $\mathrm{Vec}_G(X)$ the category of G-equivariant vector bundles on X.

If V is a G-equivariant vector bundle on X, then, for every $x \in X$, the fiber V_x carries an action of the stabilizer G_x of x in G. If $y = g \cdot x$, then $G_y = gG_xg^{-1}$ and the representation V_y of G_y can be obtained from V_x by conjugating by g. Thus, the category of G-equivariant vector bundles is semisimple with simple objects parameterized by pairs (x,σ) where x is a representative of a G-orbit on X and σ is an irreducible character of the stablizier G_x of x in G.

We consider the action of G on itself by conjugation. The set of irreducible objects of $\mathrm{Vec}_G(G)$ are parametrized by the set $\mathcal{M}(G)$ form above; we write V_m for the irreducible vector bundle corresponding to $m = (x,\sigma) \in \mathcal{M}(G)$. Moreover, we have:

Lemma 3.2. *The category* $\mathrm{Vec}_G(G)$ *is equivalent to the product of the categories* $\mathsf{Rep}(C_G(g))$, *as* g *runs over representatives of the conjugacy classes in* G.

Next we consider the Grothendieck group $K_0(\mathrm{Vec}_G(G))$ of $\mathrm{Vec}_G(G)$. Given an object V in $\mathrm{Vec}_G(G)$ and a pair $(x,y) \in G^2$ with $[x,y] = 1$, we obtain thee following two functions:

$$(x,y) \mapsto \mathrm{trace}(y, V_x), \qquad (x,y) \mapsto \mathrm{trace}(x, V_y)$$

These give rise to isomorphisms:

$$\psi_1 : K_0(\mathrm{Vec}_G(G)) \to Z_G^{(2)}, \qquad \psi_2 : K_0(\mathrm{Vec}_G(G)) \to Z_G^{(2)}.$$

The first is the same as the one arising from the equivalence of categories in Lemma 3.2, together with the ismorphism ϕ_1 from above.

Let $[V_m] \in K_0(\mathrm{Vec}_G(G))$ be the class of the irreducible vector bundle corresponding to $m = (x, \sigma) \in \mathcal{M}(G)$, and set $v_m := \phi_2^{-1} \circ \psi_2([V_m])$. The elements $\{v_m\}_{m \in \mathcal{M}(G)}$ form a basis of $\bigoplus_{g \in \frac{G}{G}} Z_{C_G(g)}$. On the other hand, for every $m' \in \mathcal{M}(G)$, we have a central character

$$\lambda_{m'} : \bigoplus_{g \in \frac{G}{G}} Z_{C_G(g)} \to \mathbb{C}$$

Evaluating, we obtain:

$$\lambda_{m'}(v_m) = \frac{|C_G(x')|}{\sigma'(1)} \{(x, \sigma), (x', \sigma')\}$$

where $\{(x, \sigma), (x', \sigma')\}$ is as from the change-of-basis from above.

4. Appendix: The (usual) Fourier transform for finite groups

Definition 4.1. The Fourier transform of $f \in \mathbb{C}[G]$ at a representation $\rho : G \to \mathrm{GL}(V)$ of G is defined as:

$$\hat{f}(\rho) = \sum_{a \in G} f(a)\rho(a) \in \mathrm{End}(V)$$

Lemma 4.2. *We have the following:*

(1) *The Fourier transform extends to an algebra homomorphism:* $F : \mathbb{C}[G] \to \mathrm{End}(V)$
(2) *(The inverse Fourier transform.) For any* $f \in \mathbb{C}[G]$ *and* $a \in G$*:*

$$f(a) = \frac{1}{|G|} \sum_{i=1}^{r} \dim(V_i) \mathrm{Trace}\left(\rho_i(a^{-1})\hat{f}(\rho_i)\right)$$

(3) *(The Plancherel Formula.) For any* $f, g \in \mathbb{C}[G]$*:*

$$\sum_{a \in G} f(a^{-1})g(a) = \frac{1}{|G|} \sum_{i=1}^{r} \dim(V_i) \mathrm{Trace}\left(\hat{f}(\rho_i)\hat{g}(\rho_i)\right)$$

These all come down to the fact that the regular representation of G decomposes into a direct sum of the representations V_i, each appearing $\dim(V_i)$ times. Consequently, the sum

$$\sum_{i=1}^{r} \dim(V_i) \mathrm{Trace}\left(\rho_i(g)\right)$$

is equal to $|G|$ if $g = e$ and zero otherwise.

4.1. The abelian case.

Suppose G is a finite abelian group. Let $\hat{G} = \mathrm{Hom}(G, S^1)$ be the set of irreducible characters of G. Then the Fourier transform and its inverse is given by:

$$\hat{f}(\chi) = \sum_{a \in G} f(a)\overline{\chi}(a)$$

Lemma 4.3. *The inverse Fourier transform gives an algebra isomorphism:*

$$F^\dagger : \mathbb{C}[\hat{G}] \to \mathbb{C}[G]$$

$$g(a) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} g(\chi)\chi(a)$$

where the source $\mathbb{C}[\hat{G}]$ *carries pointwise multiplication and* $\mathbb{C}[G]$ *carries convolution.*

Consider $F \otimes F^\dagger : \mathbb{C}[G] \otimes \mathbb{C}[\hat{G}] \to \mathbb{C}[\hat{G}] \otimes \mathbb{C}[G]$. We have that:

$$\delta_a \otimes \delta_\phi \mapsto \frac{1}{|G|} \sum_{\psi \in \hat{G}, b \in G} \psi(a)\psi \otimes \overline{\phi}(b) b.$$

Thus, as a $|G|^2$ by $|G|^2$ matrix, the linear map $F \otimes F^\dagger$ is given by:

$$\{(a,\phi),(b,\psi)\} = \frac{1}{|G|}\psi(a)\overline{\phi}(b)$$

4.2. **The cyclic case.** Let $n \geq 1$ be a positive integer and $G = \langle x \mid x^n = 1 \rangle$ be the cyclic group of order n. Let $\chi : G \to \mathbb{C}^\times$ be the character taking the generator x to the primitive n-th root of unity $\zeta = e^{\frac{2\pi i}{n}}$. The dual group $\hat{G}$ of G is generated by χ, and admits the presentation $\hat{G} = \langle \chi \mid \chi^n = 1 \rangle$. We identify the group algebra of G with the quotient $\mathbb{C}[t]/(t^n - 1)$ of the polynomial algebra by the ideal generated by $t^n - 1$. Similarly, we identify the group algebra of $\hat{G}$ with $\mathbb{C}[s]/(s^n - 1)$. Under these identifications, the Fourier and inverse Fourier transforms are given by:

$$F : \mathbb{C}[t]/(t^n - 1) \longrightarrow C[s]/(s^n - 1)$$

$$f(t) \mapsto \hat{f}(s) = \sum_{k=0}^{n-1} f(\zeta^{n-k}) s^k$$

$$F^\dagger : \mathbb{C}[s]/(s^n - 1) \longrightarrow C[t]/(t^n - 1)$$

$$g(s) \mapsto \hat{g}(t) = \frac{1}{n}\sum_{j=0}^{n-1} g(\zeta^j) t^j$$