NOTES ON KAN EXTENSIONS

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Contents

1. Introduction 1
2. Preliminaries 1
3. Kan extensions 5
4. Examples 6
References 10

1. Introduction

This expository note on Kan extensions is intended as a reference for readers with basic knowledge of category theory. The main reference is [Rie17].

2. Preliminaries

2.1. **Slice categories.** Let C be a category. Recall that the *slice category over* an object c in C, denoted C/c, is defined by:

- Objects: pairs (x, f) where c is an object of C and $f : x \to c$ is a morphism in C.
- Morphisms: a morphism $(x, f) \to (x', f')$ in the slice category is a morphism $g : x \to x'$ in C such that $f' \circ g = f$.

Similarly, we have the *slice category under* an object of C, denoted c/C, where objects are pairs (x, f) with x and object of C and $f : c \to x$ a morphism in C.

2.2. **Category of elements.** Slice categories are a special case of a more general construction, which we now describe. Let $F : C \to \mathrm{Set}$ be a functor. The *category of elements* of F, denoted $\int F$, is the category with:

- Objects: pairs (c, x) where c is an object of C and x is an element of the set Fc.
- Morphisms: a morphism $(c, x) \to (c', x')$ in the category of elements $\int F$ is a morphism $f : c \to c'$ in C such that $Ff(x) = x'$.

One can regard F as defining a set of structures on each object of the category C, and the category $\int F$ is a category of "structured objects" formed from C by selecting a choice of structure for each object. Morphisms in $\int F$ preserve the structure.

Example 2.1. The category of elements $\int C(c,-)$ of the covariant representable functor defined by c is the slice category c/C under c. Similarly, the category of elements $\int C(-,c)$ of the contravariant representable functor is the slice category C/c over c.

One can recover c as the initial object of the slice category under c; a category of elements having this property implies that it is a slice category. More precisely:

Lemma 2.2. *A functor $F : C \to \mathsf{Set}$ is representable if and only if its category of elements has an initial object.*

Proof. The proof is a straightforward application of the Yoneda lemma. □

Example 2.3. Let $K : C \to D$ be a functor and d an object of D. The *left fiber* and *right fiber* categories over d are defined as, respectively:

$$K \downarrow d = \int D(K-,d) \qquad d \downarrow K = \int D(d,K-)$$

These are also known as *comma categories*. The functor K has a left adjoint if and only if each right fiber category has an initial object. Dually, the functor K has a right adjoint if and only if each left fiber category has a terminal object.

2.3. **Limits and colimits: definitions.** Given categories J and C, a J-shaped *diagram* in a category C is a functor $F : J \to C$. These form a category, denoted C^J, where morphisms are natural transformations. We have the constant diagram functor $\Delta : C \to C^J$ taking an object c to the constant diagram with value c.

Definition 2.4. A *cone over* a J-shaped diagram $F : J \to C$ with *apex* $c \in C$ is a natural transformation $\lambda : \Delta c \to F$ from the constant diagram valued at c to F. Denoting by $\mathsf{Cone}(c, F)$ the set of such cones, we have:

$$\mathsf{Cone}(c,F) := C^J(\Delta c, F)$$

Treating the object of c as a variable, we have the functor $\mathsf{Cone}(-,F) : C^{\mathsf{op}} \to \mathsf{Set}$. The *category of cones* over F, denoted Cone/F, is defined as the category of elements of this functor:

$$\mathsf{Cone}/F := \int \mathsf{Cone}(-,F) = \int C^J(\Delta-,F)$$

A *limit* of F, if it exists, is defined via either of the following equivalent descriptions:

- A representation for the functor $\mathsf{Cone}(-,F)$, so that:
$$C(-,\lim F) \cong \mathsf{Cone}(-,F)$$
- A terminal object of Cone/F.

A limit of $F : J \to C$ comes with a cone $\lambda : \Delta(\lim F) \to F$. This gives structure morphisms $\lambda_j : \lim F \to Fj$ mapping out from the limit to the image of any objects $j \in J$:

$$\begin{tikzcd} & \lim F \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_k"] & \\ Fj \arrow[rr, "Ff"] & & Fk \end{tikzcd}$$

Key examples of limits include the product, intersection, kernel, equalizer, pullback, and inverse limit. Before examining properties of limits, we introduce the dual notion, namely colimits.

Definition 2.5. A *cone under* a J-shaped diagram $F : J \to C$ with *nadir* $c \in C$ is a natural transformation $\lambda : F \to \Delta c$ from F to the constant diagram valued at c. Denoting by $\mathsf{Cone}(F, c)$ the set of such cones, we have:

$$\mathsf{Cone}(F, c) := C^J(F, \Delta c)$$

Treating the object of c as a variable, we have the functor $\mathsf{Cone}(F, -) : C \to \mathsf{Set}$. The *category of cones* under F, denoted F/Cone, is defined as the category of elements of this functor:

$$F/\mathsf{Cone} := \int \mathsf{Cone}(F, -) = \int C^J(F, \Delta -)$$

A *colimit* of F, if it exists, is defined via either of the following equivalent descriptions:

- A representation for the functor $\mathsf{Cone}(-, F)$, so that:
$$C(\mathrm{colim} F, -) \cong \mathsf{Cone}(F, -)$$
- An initial object of F/Cone.

A colimit of $F : J \to C$ is equipped with a cone $\lambda : F \to \Delta \lim_J F$. This gives structure morphisms $\lambda_j : Fj \to \mathrm{colim}\ F$ mapping into the colimit from the image of any object $j \in J$:

$$\begin{tikzcd}
Fj \arrow[rr, "Ff"] \arrow[dr, "\lambda_j"'] & & Fk \arrow[dl, "\lambda_k"] \\
& \mathrm{colim}\ F &
\end{tikzcd}$$

Key examples of colimits include the coproduct, union, cokernel, coequalizer, pushout, and direct limit.

2.4. **Limits and colimits: properties.** We now collect results about limits and colimits.

Definition 2.6. A category C is *complete* if it contains all limits of small diagrams, i.e. diagrams $F : J \to C$ where J is a small category. A functor $C \to D$ is *continuous* if it preserves limits of small diagrams. Dually, a category C is *cocomplete* if it contains all limits of small diagrams. A functor $C \to D$ is *cocontinuous* if it preserves colimits of small diagrams.

Lemma 2.7. *If a limit exists for all functors in C^J, then a choice of limit defines a right adjoint* $\lim : C^J \to C$ *to the constant diagram functor $\Delta : C \to C^J$. Dually, if a colimit exists for all functors in C^J, then a choice of colimit defines a left adjoint* $\mathrm{colim} : C^J \to C$ *to the constant diagram functor $\Delta : C \to C^J$.*

Proof. The result follows from straightforward considerations building on the comments about fiber categories given in Example 2.3. $\square$

Lemma 2.8. *If J has an initial object, then a limit of any diagram $F : J \to C$ exists: take* $\lim F$ *to be the value of F at the initial object of J. Dually, if J has a terminal object, then a colimit of any diagram $F : J \to C$ exists: take* $\operatorname{colim} F$ *to be the value of F at the terminal object of J.*

Moreover, the identity functor $C \to C$ admits a limit (resp. colimit) if and only if C has an initial object (resp. terminal object).

Proof. The proof is a straightforward application of definitions. □

Lemma 2.9 (Right adjoints perserve limits)**.** *Let $F : C \rightleftharpoons D : G$ be an adjunction. Let $H : J \to D$ be a diagram in D so that $GH : J \to C$ is a diagram in C. If the limits of these diagrams exist, then the right adjoint G sends one to the other:*

$$G(\lim H) \simeq \lim(GH)$$

Proof. We use the Yoneda lemma and the adjunction (F, G):

$$\begin{aligned} C(c, G \lim H) &\simeq D(Fc, \lim H) = \mathrm{Cone}_D(Fc, H) \\ &= D^J(\Delta Fc, H) \simeq C^J(\Delta c, GH) \\ &= \mathrm{Cone}_C(c, GH) = C(c, \lim GH) \end{aligned}$$

The identification $D^J(\Delta Fc, H) \simeq C^J(\Delta c, GH)$ may not be obvious at first glance, but is easily verified using the adjunction (F, G). □

Dually, we have:

Lemma 2.10 (Left adjoints perserve colimits)**.** *Let $F : C \rightleftharpoons D : G$ be an adjunction. Let $H : J \to C$ be a diagram in D so that $FH : J \to D$ is a diagram in D. If the colimits of these diagrams exist, then the left adjoint F sends one to the other:*

$$F(\operatorname{colim} H) \simeq \operatorname{colim}(FH)$$

Lemma 2.11. *[Functoriality] Suppose we have a commutative diagram:*

$$\begin{tikzcd} J \arrow[rr, "G"] \arrow[dr, "F"'] & & J' \arrow[dl, "F'"] \\ & \mathcal{C} & \end{tikzcd}$$

Then, assuming the limits and colimits exist, there are canonical maps

$$\lim F' \to \lim\ F \qquad \operatorname{colim} F \to \operatorname{colim} F'$$

Proof. We can pull back a cone over F' to one over F. Specifically, if $\lambda : \Delta c \to F'$ is a cone over F' with apex c, then $j \mapsto \lambda_{Gj}$ defines a cone over F with apex c. Since the limit represents cones, we apply the Yoneda lemma to obtain a map $\lim F' \to \lim F$. Similar remarks apply for colimits. □

3. Kan extensions

We arrive to the main section of this note, wherein we introduce Kan extensions. There is a natural directional split here between left and right Kan extensions reflecting the duality between limits and colimits, which in turn reflects the directional nature of morphisms in a category.

3.1. **Definitions.** Let $K : C \to D$ be a functor and E a category. We have a pullback functor:

$$K^* : E^D \to E^C$$

which takes a functor $G : D \to E$ to its precomposition with K, so that $K^*G = G \circ K$. On the other hand, given a functor $F : C \to E$, it is less straightforward to extend F to D, i.e., to find a functor $G : D \to E$ making the following diagram commute up to a natural transformation:

$$\begin{tikzcd} C \arrow[r, "K"] \arrow[d, "F"'] & D \arrow[dl, dashed, "G"] \\ E & \end{tikzcd}$$

We have choices. The **left option** is:

- Express the commutativity relation as a natural transformation $F \to G \circ K = K^*G$, so that we consider the fiber category $F \downarrow K^*$.
- An initial object of the category $F \downarrow K^*$, if it exists, is called the *left Kan extension* of F under K, denoted $\mathrm{Lan}_K F$. It comes equipped with a natural transformation $F \to K^*\mathrm{Lan}_K F$.
- The initiality of the left Kan extension in $F \downarrow K^*$ provides a natural isomorphism.

$$E^D(\mathrm{Lan}_K F, -) \to E^C(F, K^*-)$$

- When it exists, the left Kan extension of F is given by the formula:

$$(\mathrm{Lan}_K F)(d) = \mathrm{colim}\left(K \downarrow d \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} E\right)$$

  where $\pi : K \downarrow d \to C$ is the canonical projection functor.
- If the left Kan extension exists for all $F : C \to E$, then it defines a left adjoint $\mathrm{Lan}_K : E^C \to E^D$ to the pullback functor K^*.

Dually, the **right option** is:

- Express the commutativity relation as a natural transformation $G \circ K = K^*G \to F$, so that we consider the fiber category $K^* \downarrow F$.

- A terminal object of the category $K^* \downarrow F$, if it exists, is called the *right Kan extension* of F under K, denoted $\mathrm{Ran}_K F$. It comes equipped with a natural transformation $K^*\mathrm{Ran}_K F \to F$.

- The terminal nature of the right Kan extension in $K^* \downarrow F$ provides a natural isomorphism:
$$E^D(-, \mathrm{Ran}_K F) \to E^C(K^*-, F)$$

- When it exists, the right Kan extension of F is given by the formula:
$$(\mathrm{Ran}_K F)(d) = \lim \left(d \downarrow K \xrightarrow{\pi} \mathcal{C} \xrightarrow{F} E \right)$$
where $\pi : d \downarrow K \to C$ is the canonical projection functor.

- If the right Kan extension exists for all $F : C \to E$, then it defines a right adjoint $\mathrm{Ran}_K : E^C \to E^D$ to the pullback functor K^*.

The adjunctions (when all Kan extensions exist), can be summarized in the following diagram:

$$\begin{tikzcd} E^D \arrow[r, "K^*"] & E^C \arrow[l, bend right, "\mathrm{Lan}_K"'] \arrow[l, bend left, "\mathrm{Ran}_K"] \end{tikzcd}$$

Remark 3.1. Let $g : d \to d'$ be a morphism in D. Then we have functors $K \downarrow d \to K \downarrow d'$ and $K \downarrow d' \to K \downarrow d$, commuting with the projections to C. By Lemma 2.11, taking colimits is covariant, while taking limits is contravariant, so we get a map in the correct direction in both cases.

4. Examples

4.1. **First examples.** If K is the identity functor, then K^* and Ran_K and Lan_K are all the identity functor. If $K : J \to 1$ is the functor to a point and $F : J \to C$, then $\mathrm{Lan}_K F$ and $\mathrm{Lan}_K F$ compute the colimit and limit of F, respectively, if they exist.

4.2. **Group actions.** For a group G, a G-set is a functor $\bullet/G \to \mathsf{Set}$ from the one-point category defined by G to Set; we denote this category of functors as $\mathsf{Set}(G)$. Let H be a subgroup of G. We have the inclusion of one-point categories $\bullet/H \to \bullet/G$. Pullback along this functor defines the restriction functor $\mathrm{Res} : \mathsf{Set}(G) \to \mathsf{Set}(H)$. The left and right adjoints are, respectively:
$$\mathrm{Ind}(A) = \mathrm{Lan}_{\mathrm{Res}}(A) = (G \times A)/H$$
$$\mathrm{Coind}(A) = \mathrm{Ran}_{\mathrm{Res}}(A) = \mathsf{Set}(H)(G, A)$$
To clarify: $(G \times A)/H$ is the quotient of $G \times A$ by the action of H given by $h \cdot (g, a) = (gh^{-1}, h \cdot a)$ for $g \in G$, $a \in A$, and $h \in H$; its action of G is given by $g' \cdot (g, a) = (g'g, a)$.

A morphism $f : G \to A$ in $\mathsf{Set}(H)$ is a morphism of sets such that $f(hg) = h \cdot f(g)$ for any $g \in G$ and $h \in H$; the action of G on such morphisms is $(g' \cdot f)(g) = f(g(g')^{-1})$. We have:

$$\begin{tikzcd} \mathsf{Set}(G) \arrow[r, "\mathrm{Res}"] & \mathsf{Set}(H) \arrow[l, bend right=60, "\mathrm{Ind}"'] \arrow[l, bend left=60, "\mathrm{Coind}"] \end{tikzcd}$$

In particular, suppose H is the trivial group. Then $\mathsf{Set}(H) = \mathsf{Set}$ and $\mathrm{Ind}(A) = G \times A$ is the coproduct of G-many copies of A, while $\mathrm{Coind}(A) = \mathsf{Set}(G, A) = A^G$ is the product of G-many copies of A.

4.3. **Power sets.** Let Bool be the category with two objects 0 and 1 and a single non-identity morphism $0 \to 1$. Let X be a set, regarded as a small discrete category, and let $\mathcal{P}(X)$ be the power set, regraded as a poset category under inclusion. There is an equivalence of categories:

$$\mathsf{Bool}^X \to \mathcal{P}(X)$$

taking $F : X \to \mathsf{Bool}$ to the set $A_F = \{x \in X \mid Fx = 1\}$. For functors F and G in Bool^X, there is a natural transformation $F \to G$ if and only if $A_F \subseteq A_G$. Limits in $\mathcal{P}(X)$ are intersections, while colimits are unions.

Now let $f : X \to Y$ be a map of sets, regarded as small discrete categories. There is a pullback functor $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ taking a subset $B \subseteq Y$ to $\{x \in X \mid f(x) \in B\}$. Its left and right adjoints are given by, respectively:

$$\begin{aligned} f_*(A) &= \mathrm{Lan}_{f^{-1}}(A) = \{y \in Y \mid \exists x \in f^{-1}(y) : x \in A\} \\ f_!(A) &= \mathrm{Ran}_{f^{-1}}(A) = \{y \in Y \mid \forall x \in f^{-1}(y) : x \in A\} \end{aligned}$$

In other words, $f_*(A)$ is the direct image of A consisting of all elements of the form $f(x)$ for $x \in A$. By contrast, $f_!(A)$ is the shriek image of A, consisting of all elements of y whose fibers are contained in A. We have the diagram:

$$\begin{tikzcd} \mathcal{P}(Y) \arrow[r, "f^{-1}"] & \mathcal{P}(X) \arrow[l, bend right=60, "f_*"'] \arrow[l, bend left=60, "f_!"] \end{tikzcd} \tag{4.1}$$

The adjunctions (f_*, f^{-1}) and $(f^{-1}, f_!)$ reflect the following properties:

$$\begin{aligned} f_*(A) \subseteq B \quad &\Leftrightarrow \quad A \subseteq f^{-1}(B) \\ f^{-1}(B) \subseteq A \quad &\Leftrightarrow \quad B \subseteq f_!(A) \end{aligned}$$

4.4. **Dependent sum and product.** We again regard a set X as a small discrete category. There is an equivalence of categories:

$$\mathsf{Set}^X \xrightarrow{\sim} \mathsf{Set}/X \tag{4.2}$$

Taking a functor $F : X \to \mathsf{Set}$ to the coproduct of the sets Fx, with coprojection map to X; that is, $\sum_{x\in X} Fx \to X$. We write an object of Set/X as a coproduct $S = \sum_{x\in S} S_x$. Limits and colimits in Set/X are evaluated pointwise[1].

Let $f : X \to Y$ be a map of sets. There is a pullback functor $\Delta_f : \mathsf{Set}/Y \to \mathsf{Set}/X$ given on objects by:

$$T = \sum_{y\in Y} T_y \qquad \mapsto \qquad T \times_Y X = \sum_{x\in X} T_{f(x)}$$

The definition of the functor on morphisms is straightforward. Its left adjoint is the dependent sum functor, and its right adjoint is the dependent product functor; these correspond to Kan extensions under the equivalence 4.2, and are given by:

$$\Sigma_f = \mathrm{Lan}_{\Delta_f} : \qquad \sum_{x\in X} S_x \qquad \mapsto \qquad S = \sum_{y\in Y}\left(\sum_{x\in f^{-1}(y)} S_x\right)$$

$$\Pi_f = \mathrm{Ran}_{\Delta_f} : \qquad \sum_{x\in X} S_x \qquad \mapsto \qquad \sum_{y\in Y}\left(\prod_{x\in f^{-1}(y)} S_x\right)$$

(Again, the definitions of these functors on morphisms are clear.) We have:

$$\begin{tikzcd} Y/\mathsf{Set} \arrow[rr, "\Delta_f"] & & X/\mathsf{Set} \arrow[ll, bend right, "\Sigma_f"'] \arrow[ll, bend left, "\Pi_f"] \end{tikzcd} \tag{4.3}$$

As an aside, we remark on the definition of polynomial functors. A *polynomial span* is a diagram in Set of the form $V \overset{h}{\leftarrow} A \overset{g}{\to} E \overset{t}{\to} W$, and it gives rise to a functor on slice categories given by:

$$\Sigma_t \circ \Pi_g \circ \Delta_h : \mathsf{Set}/V \to \mathsf{Set}/W$$

This is known as a *polynomial functor*, and has the explicit form:

$$\Sigma_t \circ \Pi_g \circ \Delta_h(S) = \sum_{w\in W} \sum_{e\in E_w} \prod_{a\in A_e} S_{h(a)} = \sum_{w\in W} \sum_{e\in E_w} \prod_{v\in V} S_v^{A_e \cap A_v}$$

where $A_e = g^{-1}(e)$, $A_v = h^{-1}(v)$, and $E_w = t^{-1}(w)$. One can take the definition of polynomial functors as those arising from polynomial spans. Alternatively, polynomial endofunctors on Set can be defined as the smallest class of functors containing the identity functor and constant functors, and closed under taking arbitrary coproducts and constant exponentials. See [Jac16, Section 2.2].

[1]For every $x \in X$, there is a functor $\mathrm{ev}_x : \mathsf{Set}/X \to \mathsf{Set}$ which picks out the fiber over x. It has a left and right adjoint given by the skyscraper functor $\mathsf{Set} \to \mathsf{Set}/X$ sending a set to itself with map to X factoring through the inclusion $\{x\} \hookrightarrow X$. Thus, the functor ev_x commutes with limits and colimits.

4.5. **Relational algebra.** The dependent sum and product constructions of the previous example appear in relational algebra as the union and join operations. To illustrate, let A be a chosen set of atomic values (for example, one can take $A = \mathbb{R}$).

Definition 4.1. Let H be a set. A *table* with header H is an object of the slice category Set/A^H over the set A^H of functions $H \to A$.

The idea behind this definition is the following. Given a function $p : X \to A^H$, we have a table whose rows are indexed by X and columns by H, with the (x, h) entry given by $p(x)(h)$. There are alternative definitions of a table that generalize better to database schemas[2], but the above definition is suitable for our current purposes.

Now we discuss combining tables. Let $H \to H_1 \times H_2$ be a map of sets; we think of identifying the images of H in the headers H_1 and H_2. In practice, H is a subset of the intersection $H_1 \cap H_2$. Then we have a restriction map:

$$f : A^{H_1} + A^{H_2} \to A^H$$

Observe that $\mathsf{Set}/(A^{H_1} + A^{H_2})$ is equivalent to $\mathsf{Set}/A^{H_1} \times \mathsf{Set}/A^{H_2}$. The union and join operations along f are functors:

$$\mathrm{Union}_f\ ,\ \mathrm{Join}_f\ :\ \mathsf{Set}/A^{H_1} \times \mathsf{Set}/A^{H_2} \to \mathsf{Set}/A^H$$

defined as:

$$\mathrm{Union}_f\,(X_1, X_2) = X_1 + X_2 \qquad\qquad \mathrm{Join}_f\,(X_1, X_2) = X_1 \times_{A^H} X_2$$

More explicitly, the union of $p_1 : X_1 \to A^{H_1}$ and $p_2 : X_2 \to A^{H_2}$ along f is the coproduct $X_1 + X_2$ with map to A^H given by $f \circ (p_1 + p_2)$. Meanwhile, their join along f is the pullback $X_1 \times_{A^H} X_2$ along the maps $X_i \to A^{H_i} \to A^H$ where the first map is p_i and the second map is the restriction. Special cases to consider are when H is the empty set, and when $H = H_1 = H_2$. We note that Union_f coincides with the dependent sum Σ_f, which is a left Kan extension, as noted above. Similarly, Join_f coincides with the dependent product Π_f, and is a right Kan extension.

On the other hand, the functor Δ_f sends a table $X \to A^H$ with header H to a pair of tables X_1 and X_2 with headers H_1 and H_2; one shows that X_i is the set of all possible ways (without repetition) to extend X to a table H_i and is given by $X_i = A^{H_i} \times_{A^H} X \to A^{H_i}$ for $i = 1, 2$. In the special case that $H = H_1 = H_2$, the functor Δ_f produces two copies of X.

For more on this categorical perspective on databases, see [FS19, Spi14a, Spi14b, Spi13].

4.6. **(Co)Equalizers.** Consider the category C with two objects a and b and two non-identity maps $f, g : a \to b$. A Set-valued functor X is a set X_a with two maps to a set X_b. There is a functor $F : C \to \mathsf{Bool}$ defined above (with a single non-identity morphism

[2]The main alternative is to define a table (really an *instance* of a table) as a Set-valued functor from the category with two objects c_0 and c_A with H-many arrows from c_0 to c_A; we require the functor to send c_A to the set A.

$0 \to 1$) sending a to 0 and b to 1. The left Kan extension is the coequalizer, while the right Kan extension is the equalizer:

$$\begin{aligned} \mathrm{Lan}_F X(0) &= X_a && \to \mathrm{Lan}_F X(1) = X_b / (X_f(x) \sim X_g(x)\ \forall x \in X_a) \\ \mathrm{Ran}_F X(0) &= \{x \in X_a \mid X_f(x) = X_g(x)\} && \to \mathrm{Ran}_F X(1) = X_b \end{aligned}$$

The maps are the obvious ones.

4.7. **Pullbacks.** Consider the category C with three objects a, a', and b and two non-identity maps $f : a \to b$ and $f' : a' \to b$. A Set-valued functor X is two sets X_a and $X_{a'}$, each with a map to a set X_b. There is a functor F from C to the category Bool sending a and a' to 0 and b to 1. The left Kan extension is the coproduct, while the right Kan extension is the pullback (fiber product):

$$\begin{aligned} \mathrm{Lan}_F X(0) &= X_a + X_{a'} && \to \mathrm{Lan}_F X(1) = X_b \\ \mathrm{Ran}_F X(0) &= X_a \times_{X_b} X_{a'} && \to \mathrm{Ran}_F X(1) = X_b \end{aligned}$$

4.8. **Pushouts.** Consider the category C with three objects a, b, and b', and two non-identity maps $f : a \to b'$ and $f' : a \to b'$. A Set-valued functor X is a set X_a with a map X_f to a set X_b and a map $X_{f'}$ to a set $X_{b'}$. There is a functor F from C to the category Bool sending a to 0 and b and b' to 1. The left Kan extension is the pushout, while the right Kan extension is the product:

$$\begin{aligned} \mathrm{Lan}_F X(0) &= X_a && \to \mathrm{Lan}_F X(1) = (X_b + X_{b'}) / (X_f(x) \simeq X_g(x)\ \forall x \in X_a) \\ \mathrm{Ran}_F X(0) &= X_a && \to \mathrm{Ran}_F X(1) = X_b \times X_{b'} \end{aligned}$$

4.9. **Density and codensity.** Given a functor $F : C \to D$, consider the left Kan extension of F along itself:

$$T = \mathrm{Lan}_F F : D \to D$$

We claim this is a comonad. To see this, first note that T is initial in the category $F \downarrow F^*$, so comes equipped with a natural transformation $\alpha : F \to TF$ of functors $C \to D$. Now, the identity endofunctor 1_D on D naturally defines an object of $F \downarrow F^*$ via the identity natural transformation $F \to F^* 1_D = F$. Similarly, the endofunctor T^2 defines an object of $F \downarrow F^*$ via the natural transformation $F \to TF \to T^2 F$ given by $T\alpha \circ \alpha$. From the initiality of T we obtain natural transformations $\eta : 1_D \to T$ and $\mu : T \to T^2$. One shows that these satisfy the axioms of a comonad on D, known as the *density comonad*. If F admits a right adjoint, the comonad of the adjunction FG coincides with the density comonad T.

Dually, given any functor $G : D \to C$, the codensity monad is the right Kan extension of G along itself: $T = \mathrm{Ran}_G G$. If G admits a left adjoint F, then $\mathrm{Ran}_G G$ is equivalent to the monad GF from the adjunction.

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