Quantum multiplicative quiver varieties at a root of unity

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June 18, 2019

Abstract

A common phenomenon when quantizing at roots of unity is the appearance of sheaves of Azumaya algebras on various moduli spaces. We focus on an instance of this phenomenon arising from the study of quiver representations. We begin by recalling the definition of multiplicative quiver varieties, which are constructed via Hamiltonian reduction along a group-valued moment map, and their quantizations. We state several results, and describe the techniques used in the proofs. Our methods are part of a general mechanism involving a version of quantum Hamiltonian reduction that relies on Lusztig's quantum Frobenius homomorphism and the notion of Poisson orders. This is joint work with D. Jordan and P. Safronov.

1 Introduction

These are lecture notes for a seminar given at the Kavli Institute for the Physics and Mathematics of the Universe in June 2019. They are based on joint work with David Jordan and Pavel Safronov [GJS19]. The talk consists of two parts. In the first part, I'll speak briefly about quivers and quiver representations, and about Azumaya algebras. In the second part, I'll formulate certain quantizations, state the main result, and highlight the main ideas behind the proof.

2 Quivers and quiver varieties

2.1 Representations of quivers

Let $Q = (V, E)$ be a quiver with fixed dimension vector $\mathbf{d} = (d_v) \in \mathbb{Z}^V_{\geq 0}$. For an edge $e \in E$, write $\alpha(e) \in V$ and $\beta(e) \in V$ for the source and target of e, respectively:

$$\overset{\alpha(e)}{\bullet} \xrightarrow{\ e\ } \overset{\beta(e)}{\bullet}$$

Definition 2.1. A representation of Q with dimension vector $\mathbf{d}$ is an element in

$$\mathrm{Mat}(Q, \mathbf{d}) = \bigoplus_{e \in E} \mathrm{Hom}\left(\mathbb{C}^{d_{\alpha(e)}}, \mathbb{C}^{d_{\beta(e)}}\right).$$

The group

$$G_{\mathbf{d}} = \prod_{v \in V} \mathrm{GL}_{d_v}$$

acts on $\mathrm{Mat}(Q, \mathbf{d})$ by changing the basis at each vertex

In other words, a representation is the assignment of a vector space of dimension d_v to every vertex v and a linear map from $\mathbb{C}^{d_{\alpha(e)}}$ to $\mathbb{C}^{d_{\beta(e)}}$ for every edge e. The space $\mathrm{Mat}(Q, \mathbf{d})$ is sometimes called the framed representation variety of Q, the 'framing' referring to the fact that we've chosen an identification with $\mathbb{C}^{d_v}$ at every vertex. Two representations of Q with dimension vector $\mathbf{d}$ are equivalent if and only if they lie in the same $G_{\mathbf{d}}$-orbit.

Remark 2.2. We will often suppress the dimension vector from the notation, as all our constructions are uniform in $\mathbf{d}$, and will abbreviate $\mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{d_{\alpha(e)}}, \mathbb{C}^{d_{\beta(e)}}\right)$ by $\mathrm{Mat}(e)$.

2.2 Quiver varieties

Quiver varieties are various sort of quotients

$$\mathrm{Mat}(Q, \mathbf{d})/G_{\mathbf{d}}$$

These lead to lots of interesting geometry and representation theory. One can take the quotient in many ways, e.g. using GIT quotients and Hamiltonian reduction, and can also enhance the framed representation variety before taking the quotient. In order the explain the quotient that motivates our work, let me introduce a few more players.

- The dual quiver is defined as $Q^\vee = (V, E^\vee)$, where we associate to each edge $e \in E$ a dual edge $\beta(e) \xrightarrow{e^\vee} \alpha(e)$ in $E^\vee$.
- The doubled quiver is $\overline{Q} = (V, \overline{E} = E \sqcup E^\vee)$.

Note that $\mathrm{Mat}(\overline{Q})$ can be naturally identified with the cotangent bundle $T^*\mathrm{Mat}(Q)$, which is in turn canonically a symplectic variety. From basic symplectic geometry, we know that there is a moment map:

$$\mu : T^*\mathrm{Mat}(Q) \to \mathrm{Lie}(G_{\mathbf{d}})^*,$$

which is in some sense dual to the infinitesimal action map, i.e. the map of Lie algebras:

$$\mathrm{Lie}(G_{\mathbf{d}}) \to \{\text{vector fields on } \mathrm{Mat}(Q, \mathbf{d})\}.$$

The inifinitesimal action map extends to an algebra homomorphism

$$U(\mathrm{Lie}(G_{\mathbf{d}})) \to D_{\mathrm{Mat}(Q)}.$$

from the universal enveloping algebra of the Lie algebra of $G_{\mathbf{d}}$ to the algebra of differential operators on $\mathrm{Mat}(Q)$.

2.3 Multiplicative quiver varieties and their q-deformation

We'll be interested in a multiplicative version of these maps. There is a group-valued moment map

$$\tilde{\mu} : \mathrm{Mat}(\overline{Q})^\circ \to G_{\mathbf{d}}$$

from a certain open subset of $\mathrm{Mat}(\overline{Q})$ to the group $G_{\mathbf{d}}$. There is a Poisson structure on this open subset, which one can think of as a degeneration of the symplectic structure on the cotangent bundle. Instead of extending to an algebra homomorphism from the universal enveloping algebra, one can from a q-deformation:

$$\mu_q : \mathcal{O}_q(G_{\mathbf{d}}) \to D_q(\mathrm{Mat}(Q))$$

from a certain quantum coordinate algebra of $G_{\mathbf{d}}$, known as the reflection equation algebra, to an algebra of q-difference operators on $\mathrm{Mat}(Q)$.

- [Crawely-Boevey and Shaw 2006, [CS06]]

  GIT Hamiltonian reduction along $\tilde{\mu}$ $\quad\rightsquigarrow\quad$ the multiplicative quiver variety $\mathcal{M}_Q$

- [Jordan 2014, [Jor14]]

  Quantum Hamiltonian reduction along μ_q
  $\quad\rightsquigarrow\quad$ the quantum multiplicative quiver variety $\mathcal{O}_q(\mathcal{M}_Q)$

- [G, Jordan, and Safronov 2019, [GJS19]]

  For q a primitive root of unity, the Azumaya property of D_q together with Frobenius GIT Hamiltonian reduction

  $\quad\rightsquigarrow\quad$ an Azumaya algebra $\mathcal{A}$ on the multiplicative quiver variety $\mathcal{M}_Q$ whose global sections are (approximately) the quantum multiplicative quiver variety $\mathcal{O}_q(\mathcal{M}_Q)$

As we will explain shortly, the Azumaya property amounts to saying that the fiber of $\mathcal{A}$ over any point in the multiplicative quiver vareity is a matrix algebra.

3 Azumaya algebras

Let A is any algebra over $\mathbb{C}$ and let $Z = Z(A)$ be its center. We can think of A as defining a sheaf of algebras over $\mathrm{Spec}(Z)$. Moreover, there is a central character map from the set of (isomorphism classes of) irreducible modules for A to the maximal spectrum $\mathrm{Specm}(Z)$ of Z,

$$\Phi : \mathrm{Irrep}(A) \to \mathrm{Specm}(Z).$$

(See the appendix for more details.) For a maximal ideal $\mathfrak{m}$ of Z, the fiber is $\Phi^{-1}(m)$ is the set of irreducible representations of the fiber $A \otimes_Z Z/\mathfrak{m}$. Thus, the representation theory of A can be reduced to the study of the spectrum of Z and the study of the representation theory of the various fibers.

Definition 3.1. We say that A is an Azumaya algebra if A is a finitely-generated projective R-module such that each fiber is a positive-dimensional matrix algebra. That is, for every $\mathfrak{m} \in \mathrm{Specm}(Z)$, there exists $n > 0$ and an isomorphism

$$A \otimes_Z Z/\mathfrak{m} \simeq \mathrm{Mat}_{n\times n}(\mathbb{C}).$$

Key diagram:

$$\begin{tikzcd}
\mathrm{Irrep}(A) \arrow[d, "\Phi"'] & \Phi^{-1}(\mathfrak{m}) \arrow[l, hook'] \arrow[d] \arrow[r, leftrightarrow] & \mathrm{Irrep}(\mathrm{Mat}_{n\times n}(\mathbb{C})) = \{pt\} \\
\mathrm{Specm}(Z) & \{\mathfrak{m}\} \arrow[l, hook']
\end{tikzcd}$$

Thus, the central character map is a bijection, and we can think of the category of A-modules as a twisted version of the category of Z-modules.

Remark 3.2. We make the following remarks:

1. This bijection extends to an equivalence of categories if and only if $A \simeq \mathrm{End}_R(P)$ for a finitely-generated projective generator P of R-mod. These are the 'Morita trivial' algebras, whereas Azumaya algebras.
2. The definitions extend easily to the case of a sheaf of algebras over a scheme over $\mathbb{C}$. That is, a sheaf of Azumaya algebras is one where all fibers over closed points are isomorphic to matrix algebras.
3. One can make sense of Azumaya algebras over any commutative ring; we have simplified definitions for expositional purposes.

Example 3.3. Let q be a primitive ℓ-th root of unity and consider the algebra

$$A = \mathbb{C}\langle x^{\pm 1}, y^{\pm 1}\rangle/(yx = qxy).$$

Show that the elements x^ℓ and y^ℓ are central in A, and the center is isomorphic to Laurent polynomials in x^ℓ and y^ℓ, i.e.

$$Z(A) \simeq \mathbb{C}[x^{\pm\ell}, y^{\pm\ell}].$$

Moreover, show that the algebra A is Azumaya with fibers isomorphic to $\mathrm{Mat}_{\ell\times\ell}(\mathbb{C})$.

4 Quantum multiplicative quiver varieties

I'll assume some basic familiarity with quantum groups, and I'll try to include reminders of relevant facts along the way. Feel free to ask for more detail at any point. We begin with some notation:

- Let $U_q\mathfrak{g}$ be the quantized enveloping algebra of a reductive Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Here q is any nonzero complex number. For the experts, we take the Lusztig form of the quantum group.
- Let $\mathsf{Rep}_q(G)$ b the category of locally finite-dimensional representations of $U_q\mathfrak{g}$. This is a braided tensor category in an interesting way.
- For $G = \mathrm{GL}_N$, let $R \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ be the R-matrix, which gives the braiding for $\mathbb{C}^N$ with itself as a representation of $U_q(\mathfrak{gl}_N)$. For example, for $N = 2$, we have:

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}$$

- Let $R_{21} = \tau \circ R \circ \tau$ where $\tau \in \mathrm{End}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is the flip-of-factors map.

Our aim is to define a quantum coordinate algebra $\mathcal{O}_q(\mathrm{Mat}(Q, \mathbf{d}))$ as an algebra in $\mathsf{Rep}_q(G_{\mathbf{d}})$. To this end, we first quantize $\mathrm{Mat}(M, N)$ as a representation of $\mathrm{GL}_M \times \mathrm{GL}_N$ under left and right matrix multiplication, and $\mathrm{Mat}(N, N)$ as a representation of GL_N acting by conjugation.

Definition 4.1. Define an algebra $\mathcal{O}_q(\mathrm{Mat}(M,N))$ as generated by the elements a^i_j for $i=1,\dots,N$, and $j=1,\dots,M$ subject to the following matrix relation:

$$(A\otimes \mathrm{Id}_N)(\mathrm{Id}_M\otimes A)R_{21} = R(\mathrm{Id}_N\otimes A)(A\otimes \mathrm{Id}_M)$$

where we matrix R_{21} for $\mathbb{C}^M$ appears on the left-hand side and the R-matrix for $\mathbb{C}^N$ appears on the right-hand side.

This defines an algebra in $\mathsf{Rep}_q(\mathrm{GL}_N)\boxtimes\mathsf{Rep}_q(\mathrm{GL}_M)$, known as the equivariant FRT algebra.

Example 4.2. In the case $N=1$, we recover quantum M-space, generated by variables x_j for $j=1,\dots,M$ subject to the relation:

$$x_jx_i = qx_ix_j \qquad \text{for } j<i.$$

Definition 4.3. Define an algebra $\mathcal{O}_q^{\mathrm{RE}}(\mathrm{Mat}(N,N))$ as generated by the elements a^i_j for $i,j=1,\dots,N$, subject to the following matrix relation:

$$R_{21}(A\otimes \mathrm{Id}_N)R(\mathrm{Id}_N\otimes A) = (\mathrm{Id}_N\otimes A)R_{21}(A\otimes \mathrm{Id}_N)R$$

where R is the R-matrix for $\mathbb{C}^N$.

This defines an algebra in $\mathsf{Rep}_q(\mathrm{GL}_N)$, known as the reflection equation algebra.

Example 4.4. The reflection algebra $\mathcal{O}_q^{\mathrm{RE}}(\mathrm{Mat}(2,2))$ in the case $N=2$ is generated by four elements a, b, c, and d, with relations written explicitly as:

$$\begin{array}{lll} da = ad & db = q^2bd & dc = q^{-2}cd \\ cb = bc + (1-q^{-2})(ad-d^2) & ba = ab + (1-q^{-2})bd & ca = ac + (q^{-2}-1)dc \end{array}$$

Definition 4.5. We define the following:

- Let $\mathsf{Rep}_q(G_{\mathbf{d}})$ denote the external tensor product of the categories $\mathsf{Rep}_q(\mathrm{GL}_{d_v})$ over the vertices $v\in V$. The braiding on each of the tensor factors taken together induces the structure of a braided tensor category on $\mathsf{Rep}_q(G_{\mathbf{d}})$.
- For each edge $e\in E$, set

$$\mathcal{O}_q(\mathrm{Mat}(e)) = \begin{cases} \mathcal{O}_q(\mathrm{Mat}(N,M)) \text{ if } e \text{ is not a loop: } e = \overset{N}{\bullet}\xrightarrow{e}\overset{M}{\bullet} \\ \\ \mathcal{O}_q^{\mathrm{RE}}(\mathrm{Mat}(N,N)) \text{ if } e \text{ is a loop: } e = \overset{N}{\bullet}\circlearrowleft \end{cases}$$

  These are both algebras in the $\mathsf{Rep}_q(G_{\mathbf{d}})$ where the action of $U_q(\mathfrak{gl}_{d_v})$ is trivial if v is not incident with e.
- Let

$$\mathcal{O}_q(\mathrm{Mat}(Q,\mathbf{d})) = \bigotimes_{e\in E}\mathcal{O}_q(\mathrm{Mat}(e))$$

  denote the braided tensor product of the algebras $\mathcal{O}_q(\mathrm{Mat}(e))$ in $\mathsf{Rep}_q(G_{\mathbf{d}})$.

In the definition, we invoke the following basic fact. If A and B are algebras in a braided tensor category $(\mathcal{C}, \otimes, \sigma)$, then their tensor product $A \otimes B$ carries the natural structure of an algebra with multiplication given by:

$$A \otimes B \otimes A \otimes B \xrightarrow{\sigma_{B,A}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B.$$

To be precise, the definition of $\mathcal{O}_q(\mathrm{Mat}(Q, \mathbf{d}))$ requires a choice of ordering of the edges of Q. However, the resulting algebra structure is independent of the chosen ordering.

We are also interested in quantizing the group-valued moment map

$$\tilde{\mu} : \mathrm{Mat}(\overline{Q})^\circ \to G_{\mathbf{d}}$$

In the additive setting, the right thing to do is to regard $\mathrm{Mat}(\overline{Q})$ as the cotangent bundle to $\mathrm{Mat}(Q)$ and quantize to differential operators. In the current multiplicative setting, we instead consider an algebra of q-difference operators.

$$\mu_q : \mathcal{O}_q^{\mathrm{RE}}(G_{\mathbf{d}}) \to D_q(\mathrm{Mat}(Q))$$

Here $\mathcal{O}_q^{\mathrm{RE}}(G_{\mathbf{d}})$ is the (usual, unbraided) tensor product of the algebras $\mathcal{O}_q^{\mathrm{RE}}(\mathcal{O}_q(\mathrm{GL}_{d_v}))$, where $\mathcal{O}_q^{\mathrm{RE}}(\mathcal{O}_q(\mathrm{GL}_{d_v}))$ is the localization of $\mathcal{O}_q^{\mathrm{RE}}(\mathcal{O}_q(\mathrm{Mat}(d_v, d_v)))$ at the quantum determinant $\det_q$. The algebra $D_q(\mathrm{Mat}(Q))$ is a certain smash product between $\mathcal{O}_q(\mathrm{Mat}(Q))$ and $\mathcal{O}_q(\mathrm{Mat}(Q^\vee)$, analogous to the construction of differential operators on a vector space V as the smash product of functions on V with functions on the dual V^*. We refer to the appendix for the precise relations.

Example 4.6. The easiest example is

$$D_q(\mathrm{Mat}(\overset{N}{\bullet} \xrightarrow{e} \overset{M}{\bullet})) = \mathbb{C}\langle x, \partial \rangle / \left(\partial x = q^2 x \partial + 1\right),$$

which is manifestly a q-deformation of the Weyl algebra on the affine line. The action on the polynomial algebra $\mathbb{C}[t]$ as q-difference operators is given by:

$$x \cdot f(t) = t f(t), \qquad \partial \cdot f(t) = \frac{f(q^2 t) - f(t)}{q^2 t - t}$$

Technically, we must invert the element $\alpha = 1 + (q^2 - 1)x\partial$, which has the following nice q-commutation relations:

$$\alpha x = q^2 x \alpha, \qquad \alpha \partial = q^{-2} \partial \alpha.$$

Also, if $f \in \mathbb{C}[t]$ is homogeneous of degree n, then $\alpha \cdot f(t) = q^{2n} f(t)$. Thus, α is a grading operator.

4.1 Statement of the main result

Theorem 4.7 (G., Jordan, Safronov [GJS19])**.** *Fix q to be a primitive ℓ-th root of unity, where $\ell > 1$ is odd.*

1. *There is a commutative diagram of algebra homomorphisms:*

$$\begin{tikzcd} \mathcal{O}_q^{\mathrm{RE}}(G_{\mathbf{d}}) \arrow[r, "\mu_q"] & D_q(\mathrm{Mat}(Q)) \\ \mathcal{O}(G_{\mathbf{d}}) \arrow[u] \arrow[r, "\tilde{\mu}^\#"] & \mathcal{O}(\mathrm{Mat}(\overline{Q})) \arrow[u] \end{tikzcd}$$

where the vertical maps are central embeddings.

2. *The algebra $D_q(\mathrm{Mat}(Q))$ is Azumaya over the inverse image under $\tilde{\mu}$ of the product of the big Bruhat cells in $G_{\mathbf{d}}$.*

3. *For any component-wise scalar matrix $\xi \in G_{\mathbf{d}}$ and any character $\theta : G_{\mathbf{d}} \to \mathbb{C}^\times$, there is a procedure of quantum Hamiltonian reduction that produces from $D_q(\mathrm{Mat}(Q))$ a sheaf of Azumaya algebras on the (stable) mutliplicative quiver variety corresponding to the data $(Q, \mathbf{d}, \xi, \theta)$.*

5 Techniques of the proof

5.1 Lusztig's quantum Frobenius

One technique we use is Lusztig's quantum Frobenius map. Let q be a primitive ℓ-th root of unity, where $\ell > 1$ is odd, and let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra. Then there is a 'short exact sequence of Hopf algebras':

$$u_q\mathfrak{g} \longrightarrow U_q^{\mathrm{Lus}}\mathfrak{g} \xrightarrow{\mathrm{Fr}} U\mathfrak{g}$$

where $u_q\mathfrak{g}$ is the small quantum group (a finite-dimensional Hopf algebra), $U_q^{\mathrm{Lus}}\mathfrak{g}$ is Lusztig's quantum group at the root of unity q (that is, the version with divided powers), and $U\mathfrak{g}$ is the classical enveloping algebra of $\mathfrak{g}$. The statement that this is a 'short exact sequence of Hopf algebras' means that the small quantum group includes into the Lusztig quantum group, and the quotient of the latter by the two-sided ideal generated by the augmentation ideal of the former is the classical enveloping algebra. The quotient map Fr is known as Lusztig's quantum Frobenius morphism, and leads to functors:

$$\mathrm{Fr}_* : \mathsf{Rep}_q(G) \longrightarrow \mathsf{Rep}(G)$$
$$\mathrm{Fr}^* : \mathsf{Rep}(G) \longrightarrow \mathsf{Rep}_q(G)$$

where G is an algebraic group with Lie algebra $\mathfrak{g}$, $\mathsf{Rep}(G)$ denotes the category of locally finite $U\mathfrak{g}$-modules (equivalently, the category of representations of G), and $\mathsf{Rep}_q(G)$ denotes the category of locally finite $U_q\mathfrak{g}$-modules (as above). The functor Fr_* amounts to taking invariants for the small quantum group. A key result is that the functor Fr^* is a braided tensor functor. This result, together with some direct computations with generators and relations implies the first assertion of our main result.

5.2 Poisson orders

Definition 5.1. A Poisson order is the following data:

- A commutative Poisson algebra Z.
- A central embedding of algebras $Z \hookrightarrow A$ such that the algebra A is finitely generated as a Z-module.
- A linear map
$$D : Z \longrightarrow \mathrm{Derv}(A)$$
from Z to the space of derivations of A such that $D(z_1)(z_2) = \{z_1, z_2\}$ for any z_1 and z_2 in Z.

Thus, a Poisson order is an extension of the usual action of Z on itself by derivations to an action of the (possibly non-commutative) algebra A by derivations. These arise naturally when quantizing at a root of unity.

Example 5.2. Let q be an ℓ-th root of unity (non necessarily primitive). Let

$$A = \mathbb{C}\langle x, y\rangle/(yx = qxy)$$

be the quantum plane. The elements x^ℓ and y^ℓ are central in A, and generate a central algebra isomorphic to the algebra of polynomials in x^ℓ and y^ℓ, so set $Z = \mathbb{C}[x^\ell, y^\ell]$. Endow Z with a Poisson bracket determined by $\{x^\ell, y^\ell\} = \ell x^\ell y^\ell$. The following map makes this data into a Poisson order:

$$Z \longrightarrow \mathrm{Derv}(A)$$

$$x^\ell \mapsto \begin{cases} x \mapsto 0 \\ y \mapsto x^\ell y \end{cases} \qquad y^\ell \mapsto \begin{cases} x \mapsto -xy^\ell \\ y \mapsto 0 \end{cases} \qquad .$$

The key result we need about Poisson orders is the following:

Theorem 5.3 (Brown and Gordon 2003 [BG03])**.** *Suppose (A, Z, D) is a Poisson order and $X = \mathrm{Spec}(Z)$ is a smooth variety. Let $\mathcal{L} \subseteq X$ be a symplectic leaf. Let for any two points $\mathfrak{p}$ and $\mathfrak{p}'$ of $\mathcal{L}$, there is an isomorphism of algebras:*

$$A \otimes_Z Z/\mathfrak{p} \xrightarrow{\sim} A \otimes_Z Z/\mathfrak{p}'.$$

In other words, if we regard A as defining a coherent sheaf on $X = \mathrm{Spec}(Z)$, then all fibers over a given symplectic leaf are isomorphic as algebras.

Let $G^\circ_{\mathbf{d}} \subseteq G_{\mathbf{d}} = \prod_{v\in V} \mathrm{GL}_{d_v}$ be the product of the big Bruhat cells.

Theorem 5.4 (G.-Jordan-Safronov [GJS19])**.** *The inverse image $\tilde{\mu}^{-1}(G^\circ) \subseteq \mathrm{Mat}(\overline{Q})$ is an open symplectic leaf of $\mathrm{Mat}(\overline{Q})^\circ$.*

In our paper, we show by direct computation that $D_q(\mathrm{Mat}(Q))$ is Azumaya over the point 0 in $\mathrm{Mat}(\overline{Q})$, which lies in the inverse image of the $G^\circ_{\mathbf{d}}$. Therefore, $D_q(\mathrm{Mat}(Q))$ is Azumaya over all of $\tilde{\mu}^{-1}(G^\circ)$.

5.3 Construction of the sheaf

We now explain the construction of the sheaf of Azumaya algebras on the multiplicative quiver variety.

Fix $(\xi_v)_{v\in V} \in (\mathbb{C}^\times)^V$ such that $\prod_v \xi_v = 1$ and consider the corresponding element $\xi = (\xi_v \mathrm{Id}_{d_v}) \in G_{\mathbf{d}}$. By abuse of notation, we use the same symbol to denote the corresponding map $\xi : \mathcal{O}(G_{\mathbf{d}}) \to \mathbb{C}$. Fix a lift $\overline{\xi}$ of the map ξ to $\mathcal{O}_q(G_{\mathbf{d}})$, so that we have a commutative diagram:

$$\begin{tikzcd} \mathcal{O}(G_{\mathbf{d}}) \arrow[rr, hook] \arrow[dr, "\xi"'] & & \mathcal{O}_q(G_{\mathbf{d}}) \arrow[dl, "\overline{\xi}"] \\ & \mathbb{C} & \end{tikzcd}$$

Fix a character $\theta\colon G_{\mathbf{d}} \to \mathbb{C}^\times$. Denote by $\mathbb{C}_\theta \in \mathsf{Rep}(G_{\mathbf{d}})$ the corresponding one-dimensional representation.

Definition 5.5. The multiplicative quiver variety is the GIT Hamiltonian reduction

$$\mathcal{M}(Q,\mathbf{d},\xi,\theta) = \mathcal{M}_{\mathrm{fr}}(Q)^\circ /\!\!/_{\theta} G_{\mathbf{d}} = \tilde{\mu}^{-1}(\xi)\big/_{\theta} G_{\mathbf{d}} = \mathrm{Proj}\left(\bigoplus_{m=0}^{\infty} \mathcal{O}(\tilde{\mu}^{-1}(\xi))\otimes \mathbb{C}_{\theta^{-m}}\right)^{G_{\mathbf{d}}}.$$

Let $\tilde{\mu}(\xi)^{\theta-\mathrm{ss}}$ denote the semi-stable locus of the fiber $\tilde{\mu}$ with respect to θ, and let $\tilde{\mu}(\xi)^{\theta-\mathrm{s}} \subseteq \tilde{\mu}(\xi)^{\theta-\mathrm{ss}}$ denote the stable locus. The multiplicative quiver variety $\tilde{\mu}^{-1}(\xi)\big/_{\theta} G_{\mathbf{d}}$ may be identified with the quotient of the open subset of $\tilde{\mu}^{-1}(\xi)$ of θ-semistable points of $\tilde{\mu}^{-1}(\xi)$ by a certain equivalence relation. In particular, we obtain a surjective morphism $\pi : \tilde{\mu}^{-1}(\xi)^{\theta-\mathrm{ss}} \longrightarrow \tilde{\mu}^{-1}(\xi)\big/_{\theta} G$, which fits into a commutative diagram:

$$\begin{tikzcd}
\tilde{\mu}(\xi)^{\theta-\mathrm{s}} \arrow[r, hook] \arrow[d, "\pi^s"] & \tilde{\mu}(\xi)^{\theta-\mathrm{ss}} \arrow[r, hook] \arrow[d, "\pi"] & \tilde{\mu}(\xi) \\
\mathcal{M}^s(Q,\mathbf{d},\xi,\theta) \arrow[r, hook] & \mathcal{M}(Q,\mathbf{d},\xi,\theta) &
\end{tikzcd}$$

where π^s is the restriction of π to the stable loci, and $\mathcal{M}^s(Q,\mathbf{d},\xi,\theta)$ is the image of π^s inside $\mathcal{M}(Q,\mathbf{d},\xi,\theta)$, which is known as the stable multiplicative quiver variety.

Now, observe that the fiber $\tilde{\mu}^{-1}(\xi)$ is an affine variety, given as the spectrum of the algebra $\mathcal{O}(\mathrm{Mat}(\overline{Q}))\otimes_{\mathcal{O}(G_{\mathbf{d}})} \mathbb{C}_\xi$, where $\mathbb{C}_\xi$ is the one-dimensional representation of $\mathcal{O}(G_{\mathbf{d}})$ determined by ξ. Thus, $D_q(\mathrm{Mat}(\overline{Q}))\otimes_{\mathcal{O}_q(G_{\mathbf{d}})} \mathbb{C}_{\overline{\xi}}$ defines a sheaf of algebras $\mathscr{D}_q^{\overline{\xi}}$ on $\tilde{\mu}^{-1}(\xi)$. Let $\mathscr{D}_q^{\overline{\xi},\theta}$ be the restriction of $\mathscr{D}_q^{\overline{\xi}}$ to the stable locus $\tilde{\mu}^{-1}(\xi)^{\theta-\mathrm{s}}$ of the fiber $\tilde{\mu}^{-1}(\xi)$.

Definition 5.6. Define a sheaf $\mathcal{A}$ on the stable multiplicative quiver variety as follows:

$$\mathcal{A} = \left(\pi^s_*\left(\mathscr{D}_q^{\overline{\xi},\theta}\right)\right)^{U_q\mathfrak{g}_{\mathbf{d}}}$$

Standard Hamiltonian reduction arguments show that this is a sheaf of algebras. Note that ξ is contained in $G^\circ_{\mathbf{d}} \subseteq G_{\mathbf{d}}$, the product of the big Bruhat cells in $G_{\mathbf{d}}$. Hence, $D_q(\mathrm{Mat}(Q))$ is Azumaya over $\tilde{\mu}^{-1}(\xi)$. A fiber-wise Hamiltonian reduction procedure (similar to the one that appears in [BFG06; VV10]) ensures that the fibers of $\mathcal{A}$ are also matrix algebras, and hence $\mathcal{A}$ is an Azumaya algebra.

Appendix

6 The central character map

Let A be any ring and let Z be its center. We define the central character map

$$\Phi : \mathrm{Irred}(A) \to \mathrm{Specm}(Z)$$

from the set $\mathrm{Irred}(A)$ of isomorphism classes of nonzero irreducible representations of A (i.e. simple A-modules) to the maximal spectrum of Z (i.e. maximal ideals of Z). To this end, let M be an irreducible A-module. Then the algebra of A-linear endomorphisms $\mathrm{End}_A(M)$ is a division ring. Indeed, the kernel and image of any A-linear map $f : M \to M$ is an A-submodule of M, so either f is zero, or f is invertible. The action of Z on M is A-linear, so we have an algebra homomorphism $\rho : Z \to \mathrm{End}_A(M)$. The image of this map is a field, and we set $\Phi(M) = \ker(\rho)$.

7 Azumaya algebras in general

Let R be a commutative ring.

Definition 7.1. For an R-algebra A, we say that:

1. A is a Morita trivial R-algebra if there is an equivalence of categories $A\text{-mod} \simeq R\text{-mod}$.
2. A is a Morita invertible R-algebra if there exists an R-algebra B such that $A \otimes_R B$ is a Morita trivial R-algebra.

We say that an R-module M is a generator of R-mod if $\mathrm{Hom}_R(N, M) \neq 0$ for any R-module N.

Proposition 7.2. *Let A be an R-algebra.*

1. *A is Morita trivial if and only if A is isomorphic to* $\mathrm{End}_R(P)$ *where P is a finitely generated projective R-module that is a generator of* R-mod.
2. *A is Morita invertible if and only if A is a finitely generated projective R-module that generates* R-mod*, and the following map is an isomorphism*

$$A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$$

$$a \otimes b \to [x \to axb].$$

The collection of Morita invertible R-algebras forms the Brauer group of R. These definitions extend easily to notions of sheaves of Azumaya algebras over general schemes.

Example 7.3. The quaternions $\mathbb{H}$ are Azumuaya over the reals $\mathbb{R}$.

Example 7.4. Let k be a field of characteristic $p > 0$. The Weyl algebra of the affine line over k is defined as

$$D_{\mathbb{A}^1_k} = k\langle x, \partial\rangle/([\partial, x] = 1),$$

and it is Azumaya over its center $k[x^p, \partial^p]$.

8 The algebras $\mathcal{O}_q(\mathrm{Mat}(Q))$ and $\mathscr{D}_q(\mathrm{Mat}(Q))$

The following discussion follows [Jor14, Section 4]. For positive integers $i \leq N$ and $j \leq M$, let E_i^j denote the elementary $N \times M$ matrix with 1 in the i-th row and j-th column. Let δ_j^i be the Kronecker delta symbol which evaluates to 1 if $i = j$ and 0. Define $\theta : \mathbb{Z} \to \{0,1\}$ by $\theta(k) = 1$ if $k > 0$, and 0 otherwise.

The R-matrix on $\mathbb{C}^N$ is defined as the following endomorphism of $\mathbb{C}^N \otimes \mathbb{C}^N$:

$$R := q \sum_i E_i^i \otimes E_i^i + \sum_{i \neq j} E_i^i \otimes E_j^j + \left(q - q^{-1}\right) \sum_{i > j} E_i^j \otimes E_j^i,$$

where i and j range from 1 to N. This is the universal R-matrix for the Hopf algebra $U_q(\mathfrak{gl}_N)$ and makes $\mathsf{Rep}(U_q(\mathfrak{gl}_N))$ into a braided tensor category. One can equivalently write $R = \sum_{i,j,k,l} R_{j\ell}^{ik}(E_i^j \otimes E_k^\ell)$, where

$$R_{j\ell}^{ik} = q^{\delta_k^i} \delta_j^i \delta_\ell^k + \left(q - q^{-1}\right) \theta(i-j) \delta_\ell^i \delta_j^k.$$

We will also need the the endormophism of $\mathbb{C}^N \otimes \mathbb{C}^N$ given by $R_{21} := \tau \circ R \circ \tau$, where τ is the flip of factors on $\mathbb{C}^N \otimes \mathbb{C}^N$. As a matrix, R_{21} is given by $\sum_{i,j,k,l} R_{j\ell}^{ik}(E_k^\ell \otimes E_i^j)$.

Example 8.1. In the case $N = 2$, we have

$$R = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}, \qquad \text{and} \qquad R_{21} = \begin{bmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{bmatrix}.$$

Definition 8.2. We define the following:

1. For any edge e, let $a_j^i(e)$ be formal symbols indexed by $i = 1, \ldots, d_{\alpha(e)}$ and $j = 1, \ldots, d_{\beta(e)}$. We organize the $a_j^i(e)$ into a $d_{\alpha(e)} \times d_{\beta(e)}$ matrix:

$$A^e := \sum_{i,j} a_j^i(e) E_i^j$$

2. For any edge e and vertex v, we define the following matrices:

$$A^{e,v} := A^e \otimes \mathrm{Id}_{d_v} \qquad \text{and} \qquad A^{v,e} := \mathrm{Id}_{d_v} \otimes A^e.$$

3. For $v \in V$, abbreviate by R^v the R-matrix on $\mathbb{C}^{d_v}$. For $v, w \in V$, define the matrix $R^{v,w}$ as:

$$R^{v,w} := \begin{cases} R^v & \text{if } v = w \\ \mathrm{Id}_{d_v d_w} & \text{if } v \neq w \end{cases}.$$

We have the following reformulation of Definitions 4.1, 4.3, and **??**.

Proposition 8.3. *The algebra $\mathcal{O}_q(\mathrm{Mat}(Q, \mathbf{d}))$ is generated by the elements $a_j^i(e)$ for $e \in E$, $i = 1, \ldots, d_{\alpha(e)}$, and $j = 1, \ldots, d_{\beta(e)}$ subject to the following relations.*

1. *For a non-loop edge e from v to w, i.e.* $\overset{v}{\bullet} \xrightarrow{e} \overset{w}{\bullet}$:

$$R^v A^{v,e} A^{e,w} = A^{e,v} A^{w,e} R_{21}^w.$$

2. For a loop edge at v, i.e. $\overset{v}{\bullet}\circlearrowleft$:

$$R^v_{21}A^{e,v}R^vA^{v,e} = A^{v,e}R^v_{21}A^{e,v}R^v.$$

3. For distinct edges e and f:

$$A^{f,\alpha(e)}R^{\alpha(e),\beta(f)}A^{\beta(f),e} = R^{\alpha(e),\alpha(f)}A^{\alpha(f),e}\left(R^{\alpha(f),\beta(e)}\right)^{-1}A^{\beta(e),f}R^{\beta(e),\beta(f)}.$$

Remark 8.4. The single edge relations in parts (1) and (2) of the proposition above are captured by the following single formula:

$$(R^{\alpha(e),\beta(e)})^{-1}R^{\alpha(e)}A^{\alpha(e),e}R_{21}^{\alpha(e),\beta(e)}A^{e,\beta(e)}R^{\alpha(e),\beta(e)} = R_{21}^{\alpha(e),\beta(e)}A^{e,\alpha(e)}R^{\alpha(e),\beta(e)}A^{\beta(e),e}R_{21}^{\beta(e)}(R_{21}^{\alpha(e),\beta(e)})^{-1}.$$

The two-edge cases only lead to non-trivial relations only if e and f are incident, i.e. if $\{\alpha(e),\beta(e)\}\cap\{\alpha(f),\beta(f)\}\neq\emptyset$.

Recall the dual quiver $Q^\vee$ and the doubled quiver $\overline{Q}$. Thus, we have algebras $\mathcal{O}_q(\mathrm{Mat}(Q^\vee,\mathbf{d}))$ and $\mathcal{O}_q(\mathrm{Mat}(\overline{Q},\mathbf{d}))$. Set $\Omega := \sum_{i,j} E^i_j\otimes E^j_i$.

Definition 8.5. Define an algebra $D_q^+(\mathrm{Mat}(Q,\mathbf{d}))$ as generated by the elements $a^i_j(e)$ and $\partial^j_i(e)$ for $e\in E$, $i=1,\dots,d_{\alpha(e)}$, and $j=1,\dots,d_{\beta(e)}$ subject to the following relations.

1. For a fixed edge e, the $a^i_j(e)$ generate a subalgebra ismorphic to $\mathcal{O}_q(\mathrm{Mat}(Q,\mathbf{d}))$, while the $\partial^i_j(e)$ generate a subalgebra ismorphic to $\mathcal{O}_q(\mathrm{Mat}(Q^\vee,\mathbf{d}))$ under the identification $\partial^i_j(e)\leftrightarrow a^i_j(e^\vee)$.

2. For distinct edges e and f, the $a^i_j(e)$ and $\partial^k_\ell(f)$ satisfy the same relations as do $a^i_j(e)$ and $a^k_\ell(f^\vee)$ in the algebra $\mathcal{O}_q(\mathrm{Mat}(\overline{Q},\mathbf{d}))$.

3. For a fixed edge e, the $a^i_j(e)$ and $\partial^k_\ell(e)$ satisfy the relation:

   (a) $D^{v,e}\left(R^v\right)^{-1}A^{e,v} = A^{e,w}R^wD^{w,e}+\Omega$ if e is a non-loop edge from v to w, i.e. $\overset{v}{\bullet}\xrightarrow{e}\overset{w}{\bullet}$.

   (b) $R^v_{21}D^{e,v}R^vA^{v,e} = A^{v,e}R^v_{21}D^{e,v}\left(R^v_{21}\right)^{-1}$ if e is a loop at v, i.e. $\overset{v}{\bullet}\circlearrowleft$.

The algebra $D_q^+(\mathrm{Mat}(Q,\mathbf{d}))$ can be interpreted as a smash product of $\mathcal{O}_q(\mathrm{Mat}(Q,\mathbf{d}))$ and $\mathcal{O}_q(\mathrm{Mat}(Q^\vee,\mathbf{d}))$, and has a realization as difference operators on $\mathrm{Mat}(Q)$. The algebra $D_q(\mathrm{Mat}(Q),\mathbf{d})$ we are interested in is actually a localization of $D_q^+(\mathrm{Mat}(Q),\mathbf{d})$ at certain Euler operators; we omit discussion of this localization here and refer instead to [Jor14, Section 6].

References

[BFG06] R. Bezrukavnikov, M. Finkelberg, and V. Ginzburg. "Cherednik Algebras and Hilbert Schemes in Characteristic p". In: *Representation Theory of the American Mathematical Society* 10.11 (2006), pp. 254–298.

[BG03] K. A. Brown and I. Gordon. "Poisson Orders, Symplectic Reflection Algebras and Representation Theory". In: *Journal für die Reine und Angewandte Mathematik (Crelles Journal)* 559 (2003), pp. 193–216.

[CS06] W. Crawley-Boevey and P. Shaw. "Multiplicative Preprojective Algebras, Middle Convolution and the Deligne–Simpson Problem". In: *Advances in Mathematics* 201.1 (2006), pp. 180–208.

[GJS19] I. Ganev, D. Jordan, and P. Safronov. "The Quantum Frobenius for Character Varieties and Multiplicative Quiver Varieties". In: *arXiv:1901.11450 [math]* (Jan. 2019). arXiv: `1901.11450 [math]`.

[Jor14] D. Jordan. "Quantized Multiplicative Quiver Varieties". In: *Advances in Mathematics* 250 (2014), pp. 420–466.

[VV10] M. Varagnolo and E. Vasserot. "Double Affine Hecke Algebras at Roots of Unity". In: *Representation Theory of the American Mathematical Society* 14.15 (2010), pp. 510–600.