The derived category of constructible sheaves

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1 Derived categories

Let $\mathcal{A}$ be an abelian category. This category embeds inside $\mathcal{C}(\mathcal{A})$, the category of complexes in $\mathcal{A}$. Our goal will be, in essence, to upgrade our thinking from $\mathcal{A}$ to $\mathcal{C}(\mathcal{A})$. As a category, $\mathcal{C}(\mathcal{A})$ has a few problems and so we will need to tweak and improve it into the *derived category* $\mathcal{D}(\mathcal{A})$. Though the construction is technical, there are a lot of reasons it is worthwhile:

- Derived functors like Tor^i, and Ext^i require us to take projective or injective resolutions of objects, apply a functor to whole complex, and then take i^{th} cohomology. Thus they pass through $\mathcal{C}(\mathcal{A})$. It is more natural to think of derived functors on the level of complexes.
- Passing back to $\mathcal{A}$ by taking i^{th} cohomology results in a loss of information. It would be nice to be able to keep this information and stay in $\mathcal{C}(\mathcal{A})$.
- Composing derived functions is hard in general. Computing $R^iG(R^jF(X))$ requires a spectral sequence. In the derived category, our goal is a formula like $RF \circ RG = R(F \circ G)$.
- Many operations on sheaves make more sense or can only be properly formulated on the level of complexes. For example, Grothendieck's 6 functor formalism, Riemann-Hilbert, and many operations on D-modules.

We now work towards defining $\mathcal{D}(\mathcal{A})$.

Definition. Let $\mathcal{A}$ be an abelian category. Define the category of chain complexes $\mathcal{C}(\mathcal{A})$ with

objects: $\{X^i, d^i\}_{i\in\mathbf{Z}},\ d^i : X^i \to X^{i+1},$ such that $d^{i+1} \circ d^i = 0,$
morphisms: $f^\bullet : X^\bullet \to Y^\bullet$ such that the following diagram commutes

$$\begin{array}{ccc} X^i & \xrightarrow{d_X^i} & X^{i+1} \\ \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} \\ Y^i & \xrightarrow{d_Y^{i+1}} & Y^{i+1}. \end{array}$$

The problem with this category is that it is too "fine" in the sense that functors on this category distinguish between different resolutions of the same object. In order to fix this, we want to turn quasi-isomorphism into isomorphisms. A quasi-isomorphism is a chain map

$$f : X^\bullet \longrightarrow Y^\bullet$$

which induces isomorphism on cohomology

$$H^i(f) : H^i(X^\bullet) \xrightarrow{\sim} H^i(Y^\bullet).$$

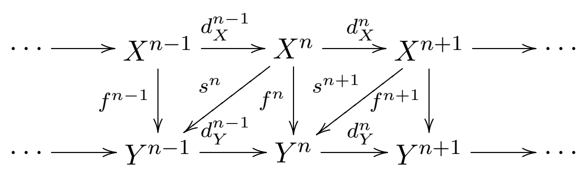
Doing this will allow us to have our cake and eat it too. We can live in the world of chain complexes without losing information by taking cohomology, but we can still consider different resolutions of the same object equivalent.

In order to formally invert the set of quasi-isomorphisms in the category, we have to localize a category similar to how one might localize a ring in commutative algebra. The complication, however, is that the hom sets are not commutative. To localize in a non-commutative setting, the hom sets must satisfy a so-called Ore condition. That is, if $S \subset A$ is a multiplicative set and we wish to multiply $a_1 s_1^{-1}$ and $a_2 s_2^{-1}$, then we need to move the a_2 past the s_1^{-1}. Hence, we need to require that there exist an $s' \in S$ and an $a' \in A$ such that

$$(a_1 s_1^{-1})(a_2 s_2^{-1}) = a_1 a' s'^{-1} s_2.$$

Our problem is that $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$ need not satisfy this condition. Thus we move to the *homotopy category* which does.

Definition. Let $f, g \in \mathrm{Hom}(X^\bullet, Y^\bullet)$. Then f is *null-homotopic*, denoted $f \sim 0$, if there exists a map $s^n : X^n \to Y^{n-1}$ such that $f = sd_X + d_Y s$. That is, in this diagram



the vertical arrow is the sum of the red and green path. In this case, we call s a *chain homotopy.* We say f is homotopic to g if $f - g \sim 0$.

Using this equivalence relation, we define the homotopy category to be a modified $\mathcal{C}(\mathcal{A})$ where morphisms are defined up to homotopy equivalence.

Definition. The homotopy category $\mathcal{K}(\mathcal{A})$ is defined with

objects: $\{X^\bullet\} = \mathrm{Ob}(\mathcal{C}(\mathcal{A}))$,
morphisms: $\mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)/\sim$.

Now we can define the derived category $\mathcal{D}(\mathcal{A})$. Very roughly, "$\mathcal{D}(\mathcal{A}) = S^{-1}\mathcal{K}(\mathcal{A})$" where S is the set of quasi-isomorphisms.

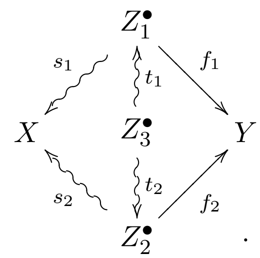
Definition. The derived category $\mathcal{D}(\mathcal{A})$ is defined with

objects: $\{X^\bullet\} = \mathrm{Ob}(\mathcal{C}(\mathcal{A}))$,
morphisms: $fs^{-1} : X^\bullet \to Y^\bullet$ is a diagram in $\mathcal{K}(\mathcal{A})$ of the form $\left[X^\bullet \overset{s}{\leftsquigarrow} Z^\bullet \xrightarrow{f} Y^\bullet \right]$, where s is a quasi-isomorphism, up to equivalence $\approx$.

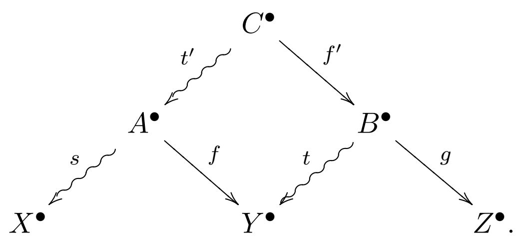
We say two diagrams are equivalent

$$\left[X^\bullet \overset{s_1}{\leftsquigarrow} Z_1^\bullet \xrightarrow{f_1} Y^\bullet \right] \approx \left[X^\bullet \overset{s_2}{\leftsquigarrow} Z_2^\bullet \xrightarrow{f_2} Y^\bullet \right]$$

if there exists $Z_3^\bullet$ and t_1, t_2 quasi-isomorphisms, such that the following diagram commutes



To compose two such morphisms $\left[X^\bullet \xleftarrow{s} A^\bullet \xrightarrow{f} Y^\bullet \right]$ and $\left[Y^\bullet \xleftarrow{t} B^\bullet \xrightarrow{g} Z^\bullet \right]$, one finds $C^\bullet, t', f'$ such the following diagram commutes



The outer wedge represents the composite morphism.

Unfortunately, the category $\mathcal{D}(\mathcal{A})$ is, in general, *not* abelian. It is, however, *triangulated*. In a triangulated category, we record a system of *distinguished triangles* which effectively remember the exact sequences from $\mathcal{C}(\mathcal{A})$. We will not list the axioms of a triangulated category here, but the most important consequence is that distinguished triangles still give rise, as did short exact sequences in an abelian category, to long exact sequences under the application of a *cohomological functor* such as H^i. Thus, although we do not have short exact sequences, we retain their most important consequences.

We can also define $\mathcal{D}^+(\mathcal{A}), \mathcal{D}^-(\mathcal{A}), \mathcal{D}^b(\mathcal{A})$ which are derived categories of $\mathcal{A}$ constructed as above but based on the full subcategories of $\mathcal{C}(\mathcal{A})$ with objects that are chain complexes bounded below, above, and fully respectively.

2 Derived category of sheaves

Goals: introduce the 6 operations on the derived category $\mathcal{D}^b(X)$ of sheaves on a topological space X, the proper base change theorem, Verdier duality, and the derived category of constructible complexes, $\mathcal{D}_c^b(X)$. The upshot will be that the category $\mathcal{D}_c^b(X)$ is closed under the 6 operations and admits a duality.

Let X be a topological space. Let $\mathsf{Shv}(X)$ be the abelian category of sheaves of $\mathbb{C}$-vector spaces on X. Let $\mathcal{D}^\#(X) = \mathcal{D}^\#(\mathsf{Shv}(X))$ where $(\# = b, +, -, \emptyset)$ depending on which bounds we impose on the complexes.

2.1 Internal Hom

The category $\mathsf{Shv}(X)$ has a left-exact internal hom functor defined as

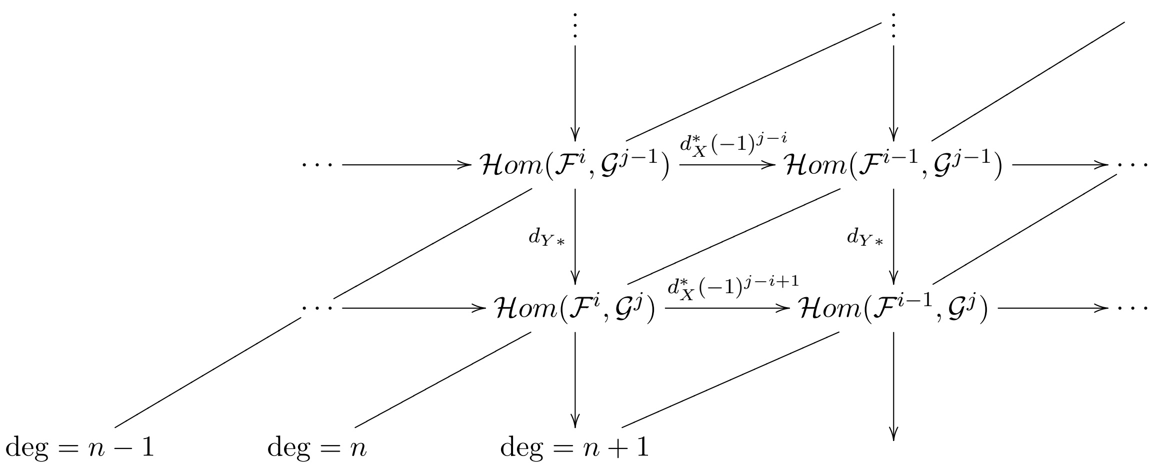
$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Since the category $\mathsf{Shv}(X)$ has enough injectives, we can define a right derived functor

$$R\mathcal{H}om : \mathcal{D}(X)^{\mathrm{op}} \times \mathcal{D}(X) \to \mathcal{D}(X)$$

as follows.

We first define a functor $\mathcal{C}(X)^{\mathrm{op}} \times \mathcal{C}(X) \to \mathcal{C}(X)$. From two complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathcal{C}(X)$, we create a double complex $\mathcal{H}om(\mathcal{F}^i, \mathcal{G}^j)$ and then take the associated total complex given by the diagonals having constant degree $\deg = j - i$. A chain map of the total complex is given by all of the boundary maps going between two adjacent diagonals signed as necessary in order to assure $d^2 = 0$. This total complex is then the image of our functor. We can diagram this as follows,



Conveniently, this functor respects chain homotopies and thus descends to

$$\widetilde{\mathcal{H}om} : \mathcal{K}(X)^{\mathrm{op}} \times \mathcal{K}(X) \to \mathcal{K}(X).$$

Now we want to induce a functor on the derived category. Briefly and intuitively, if we have a functor F between homotopy categories and we want one on the level of derived categories, then we need to apply F to a diagram such as

$$\left[X^\bullet \overset{s}{\leftsquigarrow} Z^\bullet \xrightarrow{f} Y^\bullet \right]$$

and have the resulting diagram be of the same form. Thus we would like $F(s)$ to be a quasi-isomorphism. Generally, this is not true, but it is true if we restrict to the full subcategory of injective complexes. Fortunately, since our category has enough injectives, every complex is quasi-isomorphic to an injective complex. Thus, we lose nothing in making such a restriction. So we have a functor

$$\mathcal{K}(X)^{\mathrm{op}} \times \mathcal{D}(X) \to \mathcal{D}(X)$$

defined by

$$\mathcal{F}^\bullet, \mathcal{G}^\bullet \longmapsto \widetilde{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$$

where $\mathcal{I}^\bullet$ is an injective complex chosen quasi-isomorphic to $\mathcal{G}^\bullet$. Similar to the commutative case, localization has a universal property which allows us to extend to denominators in the domain if they are already mapped to something invertible in the target. Hence we induce

$$R\mathcal{H}om : \mathcal{D}(X)^{\mathrm{op}} \times \mathcal{D}(X) \to \mathcal{D}(X).$$

By imposing bounds on the total degree and noting the implied bounds on i and j we can also get functors with a variety of bounds:

$$R\mathcal{H}om : \mathcal{D}^-(X)^{\mathrm{op}} \times \mathcal{D}^+(X) \to \mathcal{D}^+(X),$$
$$R\mathcal{H}om : \mathcal{D}^+(X)^{\mathrm{op}} \times \mathcal{D}^-(X) \to \mathcal{D}^-(X),$$
$$R\mathcal{H}om : \mathcal{D}^b(X)^{\mathrm{op}} \times \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

Let $\mathcal{F}, \mathcal{G}$ be sheaves over X, i.e. complexes concentrated in degree 0. Taking homology gives the traditional right-derived functors of $\mathcal{H}om$

$$\mathcal{E}xt^n(\mathcal{F}, \mathcal{G}) = \mathcal{H}^n(R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

2.2 Derived direct and inverse image functors

Let $f : X \to Y$ be a continuous function between topological spaces. We are interested in the sorts of functors that f induces on the derived level. Many of them start with functors on the abelian level. The direct image functor

$$f_* : \mathsf{Shv}(X) \to \mathsf{Shv}(Y)$$

is defined by $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. This is a left exact functor, and has a right derived functor

$$Rf_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y).$$

Explicitly, if $\mathcal{F}$ is a sheaf, then $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F})$. Stalks are computed using homology of preimages of neighborhoods.

Example. Let f be the inclusion of $\mathbb{C} \setminus \{0\}$ into $\mathbb{C}$. The stalk over 0 of $R^i f_*(\mathbb{C})$ is $H^i(S^1, \mathbb{C})$. Explanation: the preimage under f of any sufficiently small neighborhood of the origin in $\mathbb{C}$ is a punctured disk, which has the same cohomology as a circle.

The inverse image functor

$$f^* : \mathsf{Shv}(Y) \to \mathsf{Shv}(X)$$

is defined[1] by $f^*\mathcal{F}(U) = \lim_{V \supset f(U)} \mathcal{F}(V)$. Note that f^* behaves well with stalks: $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$. Using the fact that a sequence of sheaves is exact if and only if it is exact on stalks, we conclude that f^* is an exact functor. Therefore, the functor f^* descends to a functor on the derived category:

$$f^* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X).$$

Adjunction forumlas for $\mathcal{F} \in \mathcal{D}^b(Y)$ and $\mathcal{G} \in \mathcal{D}^b(X)$:

$$Rf_* R\mathcal{H}om^\bullet(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) = R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{\mathcal{D}^b(Y)}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet)$$

We also have the composition rule which helped motivate our discussion of derived categories.

[1] It is common to write f^{-1} for this functor and reserve f^* for pullback of sheaves of modules or coherent sheaves.

Proposition 1. *If* $F : \mathsf{Shv}(X) \to \mathsf{Shv}(Y)$ *and* $G : \mathsf{Shv}(Y) \to \mathsf{Shv}(Z)$ *and* F *preserves injectives (e.g.* $F = f_*$*) then*

$$R(G \circ F) = RG \circ RF.$$

2.3 Derived direct image with compact supports

Define the direct image with compact supports functor as

$$f_! : \mathsf{Shv}(X) \to \mathsf{Shv}(Y)$$

$$f_!\mathcal{F}(V) = \{s \in \Gamma(f^{-1}(V), \mathcal{F}) \mid f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to V \text{ is proper}\}.$$

Observe that if f is a proper map, then $f_* = f_!$. The functor $f_!$ is left exact, and we can define its derived functor

$$Rf_! : \mathcal{D}^b(X) \to \mathcal{D}^b(Y).$$

Example. Let f be the inclusion of $\mathbb{C} \setminus \{0\}$ into $\mathbb{C}$. The stalk over 0 of $R^i f_!(\mathbb{C})$ is the cohomology with compact supports $H^i_c(D^\circ, \mathbb{C})$ where D° is a punctured disk. By Poincaré duality, we have that $R^i f_!(\underline{C}_X)_0$ is equal to $\mathbb{C}$ if $i = 1, 2$ and 0 otherwise.

Why study the direct image with compact supports? There are many reasons; the ones I would like to focus on are: the proper base change theorem and Verdier duality.

2.4 Proper base change theorem and the projection formula

Theorem 2 (Proper base change theorem)**.** *Suppose we have a Cartesian diagram*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

Then $g^* \circ f_! = (f')_! \circ (g')^*$*. Moreover,* $g^* \circ Rf_! = R(f')_! \circ (g')^*$

The following example illustrates that the above theorem does not hold if the direct image with compact supports is replaced by the ordinary direct image functor.

Example. [3, Exercise 2.3.28] Let X be the unit circle in $\mathbb{R}^2$ with the point $\{(0,1)\}$ removed, i.e. $X = S^1 \setminus \{(0,1)\}$. Let Y be the interval $[-1,1]$ and $f : X \to Y$ the projection onto the first coordinate. Let $Z = \{0\}$ and $g : Z \to Y$ the inclusion of the origin in the interval. Then, for any sheaf $\mathcal{F}$ on X, $g^* \circ f_!(\mathcal{F})$ and $(f')_! \circ (g')^*(\mathcal{F})$ both compute the stalk of $\mathcal{F}$ at the point $\{(0,-1)\}$. On the other hand, taking $\mathcal{F} = \underline{\mathbb{C}}_X$ to be the constant sheaf, we have $g^* \circ f_*(\mathcal{F})(\underline{\mathbb{C}}_X) = \mathbb{C}^3$ and $(f')_* \circ (g')^*(\underline{\mathbb{C}}_X) = \mathbb{C}$.

As foreshadowing, we ask the following question: Does there exist a version of the proper base change theorem with ordinary direct images? The answer will be revealed in the section on Verdier duality.

I also want to mention the projection formula at this point. Recall that the category $\mathsf{Shv}(X)$ has a tensor product: $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf taking U to $\mathcal{F}(U) \otimes_{\mathbb{C}} \mathcal{G}(U)$. The category $\mathsf{Shv}(X)$ has enough flat objects, so there is a derived tensor product

$$- \otimes^L - : \mathcal{D}^b(X) \times \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

Theorem 3 (Projection formula). *For any $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{G}^\bullet \in \mathcal{D}^b(Y)$, we have*

$$Rf_!\mathcal{F}^\bullet \otimes^L \mathcal{G}^\bullet \simeq Rf_!(\mathcal{F}^\bullet \otimes^L f^*\mathcal{G}^\bullet).$$

2.5 Verdier Duality

Theorem 4 (Verdier duality). *If $R^k f_! = 0$ for $k >> 0$ (i.e. the functor $f_!$ finite cohomological dimension), then the functor $Rf_!$ has a right adjoint $f^!$. More precisely, for $\mathcal{F} \in \mathcal{D}^b(X)$ and $\mathcal{G} \in \mathcal{D}^b(Y)$:*

$$R\mathcal{H}om^\bullet(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Rf_*R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$
$$\mathrm{Hom}_{\mathcal{D}^b(Y)}(Rf_!\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$

Henceforth assume all maps f satisfy the condition of the above theorem. Note that the functor $f^!$ is not the the derived functor of a functor on the abelian level. This illustrates a great advantage to working with derived categories. The following result gives a more concrete sense of what $f^!$ is doing.

Proposition 5. *Suppose f is smooth with smooth fibers of dimension d. Then $f^! = f^*[2d]$.*

We have the following version of the proper base change theorem:

Proposition 6. *Suppose we have a Cartesian diagram*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

Then $g^! \circ Rf_* = R(f')_* \circ (g')^!$.

Definition. Let $a_X : X \to \mathrm{pt}$ be the unique map from X to a point. The *dualizing sheaf* of X is defined as

$$\omega_X = (a_X)^!\mathbb{C}.$$

If $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$, then define the '*dual*' of $\mathcal{F}^\bullet$ as

$$D\mathcal{F}^\bullet = R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \omega_X).$$

Properties:

- $f^!\omega_Y = \omega_X$.
- If X is smooth complex manifold, then

  $$\omega_X = \underline{\mathbb{C}}_X[2\dim_{\mathbb{C}}(X)].$$

  (The shift in dimension is the 'same' as the shift in dimension that appears in Poincaré duality. In fact, Verdier duality is a generalization of Poincaré duality.)
- The assignment D interchanges shrieks and stars, more precisely, for $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ and $\mathcal{G}^\bullet \in \mathcal{D}^b(Y)$, we have:

  $$f^!(D\mathcal{G}^\bullet) = D(f^*\mathcal{G}^\bullet) \qquad Rf_*(D\mathcal{F}^\bullet) = D(Rf_!\mathcal{F}^\bullet).$$

A problem with the assignment $\mathcal{F} \mapsto D\mathcal{F}$ is that it does not square to the identity, so is not a proper duality. In the following section, we define a subcategory of $\mathcal{D}^b(X)$ where $D^2 = \mathrm{Id}$.

2.6 Derived category of constructible complexes

To motivate the derived category of constructible complexes, we state the following fact:

Fact: Local systems are not preserved by the functors $Rf_*, Rf_!, f^*, f^!$ in general

We seek a subcategory of $\mathcal{D}^b(X)$ that includes local systems and is closed under these operations. The smallest such category (in some sense) is given in the following definition:

Definition. The (bounded) *derived category of constructible complexes*, denoted $\mathcal{D}_c^b(X)$ is the full subcategory of $\mathcal{D}^b(X)$ consisting of objects $\mathcal{F}$ such that each cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ is constructible.

The notion of a constructible sheaf was defined in the previous talk.

Theorem 7. *The category $\mathcal{D}_c^b(X)$ is closed under the 6 operations:*

$$Rf_*, f^*, Rf_!, f^!, R\mathcal{H}om^\bullet, \otimes^L.$$

Moreover, $\mathcal{D}_c^b(X)$ is closed under D, and D squares to the identity on objects of $\mathcal{D}_c^b(X)$.

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