Notes on equivariant sheaves and $\mathcal{D}$-modules

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This collection of expository notes emerged while working on the paper [BG19], and from conversations with D. Ben-Zvi, Sam Gunningham, D. Jordan, P. Safronov, T. Schedler, and others. References include [CG09,HTT07]. Unless specified otherwise, we work over $\mathbb{C}$.

1. Equivariant Sheaves

1.1. **General definitions.** Suppose a linear algebraic group G acts on a variety X. What is an equivariant sheaf on X? As motivation, we first give the definition of a G-equivariant vector bundle on X.

Definition 1.1. Suppose $p : E \to X$ be a vector bundle. We say that E is a G-equivariant vector bundle if there is an action of G on E such that the map p is equivariant, and the action is linear on fibers. In this case, the sheaf $\mathcal{E}$ of sections of E is a G-equivariant sheaf.

To be explicit, for every $x \in X$, the action of $g \in G$ defines a linear isomorphism $\phi_{(g,x)} : E_{gx} \to E_x$ from the fiber of E over gx to the fiber over x. These isomorphisms fit into a smooth family, and satisfy the associativity condition $\phi_{(h,x)} \circ \phi_{(g,hx)} = \phi_{(gh,x)}$, for any $g, h \in G$ and $x \in X$.

Definition 1.1 generalizes to arbitrary sheaves as follows (c.f. [CG09, Section 5.1]). Let

$$a : G \times X \to X \qquad\qquad p : G \times X \to X$$

be the action and projection maps, let $p_{23} : G \times G \times X \to G \times X$ be the projection onto the second and third factors, and let $m : G \times G \to G$ be the multiplication map.

Definition 1.2. An G-equivariant sheaf on X is a pair $(\mathcal{F}, \Phi)$, where $\mathcal{F}$ is a quasicoherent sheaf on X and $\Phi : a^*\mathcal{F} \to p^*\mathcal{F}$ is an isomorphism subject to the condition that the following associativity equality holds:

$$p_{23}^*\Phi \circ (1 \times a)^*\Phi = (m \times 1)^*\Phi.$$

A morphism from $(\mathcal{F}, \Phi_{\mathcal{F}})$ to $(\mathcal{G}, \Phi_{\mathcal{G}})$ is a morphism of $\mathcal{O}_X$-modules $f : \mathcal{F} \to \mathcal{G}$ such that $p^* f \circ \Phi_{\mathcal{F}} = \Phi_{\mathcal{G}} \circ a^* f$. We denote the category of G-equivariant quasicoherent sheaves on X by $\mathsf{QCoh}_G(X)$.

This definition can be easily extended to include G-equivariant sheaves of $\mathcal{O}_X$-modules, rather than just quasicoherent sheaves; however, we will focus on the quasicoherent setting. To make sense of the associativity constraint, first observe that the following diagrams commute:

$$\begin{array}{ccc} G\times G\times X & \xrightarrow{m\times 1} & G\times X \\ \downarrow{\scriptstyle 1\times a} & & \downarrow{\scriptstyle a} \\ G\times X & \xrightarrow{a} & X \end{array} \qquad \begin{array}{ccc} G\times G\times X & \xrightarrow{p_{23}} & G\times X \\ \downarrow{\scriptstyle 1\times a} & & \downarrow{\scriptstyle a} \\ G\times X & \xrightarrow{p} & X \end{array} \qquad \begin{array}{ccc} G\times G\times X & \xrightarrow{p_{23}} & G\times X \\ \downarrow{\scriptstyle m\times 1} & & \downarrow{\scriptstyle p} \\ G\times X & \xrightarrow{p} & X \end{array}.$$

Consequently, the constraint asserts that the following diagram commutes:

$$\begin{array}{ccccc} (1\times a)^* a^*\mathcal{F} = (m\times 1)^* a^*\mathcal{F} & & \xrightarrow{(m\times 1)^*\Phi} & & (m\times 1)^* p^*\mathcal{F} = p_{23}^* p^*\mathcal{F} \\ & \searrow^{(1\times a)^*\Phi} & & \nearrow_{p_{23}^*\Phi} & \\ & & (1\times a)^* p^*\mathcal{F} = p_{23}^* a^*\mathcal{F} & & \end{array}$$

1.2. **Hopf algebras.** Definition 1.2 can be restated more algebraically in the case where X is affine. In order to do so, we recall some facts about Hopf algebras and their categories of representations. We refer the reader to, e.g., [Kas12] for more details.

Let H be a Hopf algebra. Then the category H-comod of H-comodules is a tensor category. Indeed, if M and N are H-comodules, then the coaction of H on the tensor product (of vector spaces) $M \otimes N$ is given by

$$M \otimes N \xrightarrow{\text{coact}\otimes\text{coact}} (H \otimes M) \otimes (H \otimes N) \xrightarrow{\sim} (H \otimes H) \otimes (M \otimes N) \xrightarrow{\text{m}\otimes 1\otimes 1} H \otimes (M \otimes N).$$

Given an algebra object A in the tensor category H-comod, we write $A\text{-mod}_{H\text{-comod}}$ for the category of A-modules in the category H-comod. Let

$$a^\# : A \to H \boxtimes A \qquad p^\# : A \to H \boxtimes A$$

denote the coaction map and the inclusion of the second factor, respectively. We use the symbol '$\boxtimes$' to emphasize that we take the tensor product of abstract algebras, rather than the tensor product within the category of H-comodules. We have corresponding functors between categories of modules:

$$a^* : A\text{-mod} \rightleftarrows H \boxtimes A\text{-mod} : a_* \qquad p^* : A\text{-mod} \rightleftarrows H \boxtimes A\text{-mod} : p_*$$

Lemma 1.3. *An object of $A\text{-mod}_{H\text{-comod}}$ is equivalent to the data of an A-module M equipped with an isomorphism $a^*M \to p^*M$ satisfying an associativity condition analogous to the one in Definition 1.2 above.*

Sketch of proof. First, suppose that M is an A-module in H-comodules. The fact that the action map $A \otimes M \to M$ is a map of H-comodules implies (and is in fact is equivalent to) the commutativity of the following diagram, where $\beta : M \to H \boxtimes M$ is the coaction map:

$$\begin{array}{ccc}
A \otimes M & \xrightarrow{1\otimes\beta} & A \otimes (H \boxtimes M) \\
 & & \downarrow{\scriptstyle a^{\#}\otimes 1} \\
 & & (H \boxtimes A) \otimes (H \boxtimes M) \\
\downarrow{\scriptstyle \text{act}} & & \downarrow{\scriptstyle \sim} \\
 & & (H \otimes H) \boxtimes (A \otimes M) \\
 & & \downarrow{\scriptstyle \text{mult } \boxtimes \text{ act}} \\
M & \xrightarrow{\beta} & H \boxtimes M
\end{array} .$$

In turn, the commutativity of this diagram implies that β is a map of A-modules, where $H \boxtimes M$ is identified with $a_* p^* M$. The adjunction (a^*, a_*) gives an identification:

$$\mathrm{Hom}_A(M, a_* p^* M) = \mathrm{Hom}_{H\boxtimes A}(a^* M, p^* M)$$

Hence, the coaction map β corresponds to a map

$$\Phi : a^* M \to p^* M.$$

We leave it as an exercise to show that Φ is an isomorphism and the coassociativity of the coaction of H on M recovers the associativity condition of Φ. □

1.3. **Affine algebraic groups.** Let G be an affine algebraic group and let $\mathsf{Rep}(G)$ be the category of representations of G. Since G is a group, the coordinate algebra $\mathcal{O}_G$ of G is a commutative Hopf algebra.

Lemma 1.4. *There is an equivalence of categories* $\mathsf{Rep}(G) = \mathcal{O}_G$-comod.

Proof. There is a functor $\mathcal{O}_G$-comod $\to \mathsf{Rep}(G)$ is given by evaluation. The functor in the other direction is given as follows. Let $V \in \mathsf{Rep}(G)$, and fix $v \in V$. The G-orbit Gv is a finite dimensional subspace of V. Choosing a basis $\{e_i\}$ for this subspace and a dual basis $\{e^i\}$ for its dual, we define a coaction of $\mathcal{O}_G$ on V by $\Delta(v) = [g \mapsto \langle e^i, gv\rangle] \otimes e_i$. □

Lemma 1.5. *There is a fully-faithful functor*

$$H\text{-comod} \to \mathcal{U}\mathfrak{g}\text{-mod}$$

commuting with the forgetful functors to vector spaces.

Proof. There is a natural evaluation pairing $\kappa : \mathcal{O}_G \otimes \mathcal{U}\mathfrak{g} \to \mathbb{C}$ given by evaluation of matrix coefficients. Given an H-comodule M, define a $\mathcal{U}\mathfrak{g}$-action on M via the following composition:

$$\mathcal{U}\mathfrak{g} \otimes M \xrightarrow{1\otimes\text{coact}} \mathcal{U}\mathfrak{g} \otimes H \otimes M \xrightarrow{\kappa\otimes 1} M.$$

□

Remark 1.6. The Hopf algebra $\mathcal{O}_G$ is the Hopf dual of $\mathcal{U}\mathfrak{g}$, but $U\mathfrak{g}$ is not the Hopf dual of $\mathcal{O}_G$. See [BG12, Section I.9] the other way around.

1.4. **The affine case.** Let X be an affine scheme, so that $X = \mathrm{Spec}(\mathcal{O}_X)$ for a commutative algebra $\mathcal{O}_X$. The data of an action of G on X is equivalent to the data of a coaction of $H = \mathcal{O}_G$ on $\mathcal{O}_X$ such that the multiplication map $\mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ is a map of H-comodules. As a consequence of Lemma 1.3, we have:

Proposition 1.7. *The category of G-equivariant sheaves on X is equivalent to the category of $\mathcal{O}_X$-modules in the category of $\mathcal{O}_G$-comodules:*

$$\mathsf{QCoh}_G(X) \xrightarrow{\sim} \mathcal{O}_X\text{-mod}_{\mathcal{O}_G\text{-comod}}.$$

Remark 1.8. We comment on the non-commutative setting. Let $\mathcal{O}_q(G)$ be the quantum coordinate algebra of a reductive group G. This is a Hopf algebra, and so its category of comodules $\mathcal{O}_q(G)$-comod carries the structure of a tensor category. An algebra object A in the category of $\mathcal{O}_q(G)$-comodules can be regarded as a non-commutative G-variety, and the category of modules can be regarded as the category of equivariant sheaves on this (non-existent) non-commutative space. Backelin Kremnizter consider the particular case of the quantum flag variety [BK06].

1.5. **Descent.** Alternatively, one can define the category of equivariant sheaves on X as the category of sheaves on the stack X/G via descent, noting the following facts.

Definition 1.9. The action groupoid $X \times_{X/G} X$ consists of triples (x, y, g) where $x, y \in X$ and g takes x to y. We have projection maps onto the first and second factor:

$$\pi_1 : X \times_{X/G} X \to X \qquad\qquad \pi_2 : X \times_{X/G} X \to X.$$

Similarly, we have $X \times_{X/G} X \times_{X/G} X$ with the projections $\pi_{ij} : X \times_{X/G} X \times_{X/G} X \to X \times_{X/G} X$ for $(ij) \in \{(12), (13), (23)\}$.

Definition 1.10. The category of sheaves on X/G has as objects pairs $(\mathcal{F}, \Psi)$, where $\mathcal{F}$ is a quasicoherent sheaf on X and $\Psi : \pi_1^*\mathcal{F} \to \pi_2^*\mathcal{F}$ is an isomorphism of sheaves on $X \times_{X/G} X$ subject to the condition that the following associativity equality holds:

$$\pi_{23}^*\Psi \circ \pi_{12}^*\Psi = \pi_{13}^*\Psi.$$

A morphism between sheaves are defined in the obvious way.

To make sense of the associativity constraint, we use the fact that $\pi_{12}^*\pi_1^* = \pi_{13}^*\pi_1^*$, $\pi_{12}^*\pi_2^* = \pi_{23}^*\pi_1^*$, and $\pi_{13}^*\pi_2^* = \pi_{23}^*\pi_2^*$.

Lemma 1.11. *The action groupoid $X \times_{X/G} X \rightrightarrows X$ is the same as $G \times X \rightrightarrows X$, where π_1 is identifies with p and π_2 is identified with a.*

The proof of the above lemma is elementary: the isomorphism in question takes (x, y, g) to (g, x). Similarly, there is an isomorphism between $X \times_{X/G} X \times_{X/G} X$ and $G \times G \times X$ in which the projections $\pi_{ij} : X \times_{X/G} X \times_{X/G} X \to X \times_{X/G} X$ correspond to p_{23} when $i = 1, j = 2$; to $m \times 1$ when $i = 1, j = 3$; and to $1 \times a$ when $i = 2, j = 3$. The following lemma is now clear from definitions:

Lemma 1.12. *There is an equivalence of categories between the category of G-equivariant sheaves on X and the category of sheaves on the action groupoid $X \times_{X/G} X$.*

1.6. **Case of a torus.** We consider the case where $G = T$ is a torus. Let $\Lambda = X^*(T)$ be the character lattice of T. Every character $\lambda \in \Lambda$ gives rise to an algebraic function z^λ on T, and we have that $z^\lambda z^\mu = z^{\lambda+\mu}$ for $\lambda, \mu \in \Lambda$. These functions span the algebra of functions on T, which can thus be identified with the group algebra of the lattice Λ:

$$\mathcal{O}_T = \mathbb{C}[\Lambda] = \mathbb{C}[z^\lambda \mid \lambda \in \Lambda].$$

The Hopf structure on $\mathcal{O}_T$ is given by

$$\Delta(z^\lambda) = z^\lambda \otimes z^\lambda \qquad \epsilon(z^\lambda) = 1 \qquad S(z^\lambda) = z^{-\lambda},$$

for any $\lambda \in \Lambda$.

Lemma 1.13. *The category of $\mathcal{O}_T$-comodules is equivalent to the category of Λ-graded vector spaces.*

Proof. Let M be an $\mathcal{O}_T$ comodule with coaction map $\Delta : M \to \mathcal{O}_T \otimes M$. For $\lambda \in \Lambda$, define

$$M_\lambda = \{m \in M \mid \Delta(m) = z^\lambda \otimes m' \text{ for some } m' \in M\}.$$

It is clear that $M_\lambda \cap M_\mu = 0$ if $\lambda \neq \mu$, and the counit axiom implies that $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$. Conversely, given a graded vector space $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$, define a coaction of $\mathcal{O}_T$ on V by

$$\Delta : V \to \mathcal{O}_T \otimes V; \qquad v_\lambda \mapsto z^\lambda \otimes v_\lambda$$

for $v_\lambda \in V_\lambda$. □

2. Some ring theory

In this section we switch gears and discuss some ring theory. Let A be an algebra over $\mathbb{C}$.

Definition 2.1. An A-ring is a ring R equipped with an homomorphism $\tau : A \to R$. We write τ_a for the image of $a \in A$.

We give some examples of A-rings.

- The algebra A is an A-ring with τ being the identity map.
- The ring $(\mathrm{End}_\mathbb{C}(A), \circ)$ of $\mathbb{C}$-linear endomorphisms of A with the operation of composition is an A-ring with
$$\tau : A \to \mathrm{End}_\mathbb{C}(A); \qquad a \mapsto [\tau_a : b \mapsto ab].$$
- The ring $(\mathrm{End}_\mathbb{C}(A), \circ^{\mathrm{op}})$ of $\mathbb{C}$-linear endomorphisms of A with the opposite multiplication is an A-ring with
$$\tau : A \to \mathrm{End}_\mathbb{C}(A); \qquad a \mapsto [\tau_a : b \mapsto ba].$$
- The algebra $\mathrm{Diff}(A) \subseteq \mathrm{End}_\mathbb{C}(A)$ of differential operators is the subalgebra of $\mathrm{End}_\mathbb{C}(A)$ generated by the image of τ and by the derivations of A. Hence it is an A-ring.

The category A-bimod of A-bimodules is a tensor category under the tensor product $- \otimes_A -$.

Lemma 2.2. *An A-ring R defines an algebra object in the category of A-bimodules. Consequently, R defines a monad on the category A-mod via $M \mapsto R \otimes_A M$.*

Proof. The A-bimodule structure on an A-ring R is defined as follows: $a \otimes b \in A \otimes A$ takes $r \in R$ to the product $\tau_a \cdot r \cdot \tau_b$. The first statement follows from the fact that the multiplication on R factors through $R \otimes_A R \to R$, and this is a morphism of A-bimodules. For the second statement, the A-module structure on $R \otimes_A M$ comes from the remaining left A-action on the factor R. The multiplication on the monad comes from the fact the multiplication on R factors through $R \otimes_A R \to R$. The unit on the monad is given by $M \to R \otimes_A M$, $m \mapsto 1 \otimes m$. □

Remark 2.3. The category of modules for the monad on A-mod defined by R is equivalent to R-mod.

We say that R is an A-algebra if A is commutative and the image of τ is central in R. In this case, the left and right actions of A on R coincide. Moreover, the category A-mod of A-modules is a tensor category under the relative tensor product $- \otimes_A -$, and R is an algebra object in this category.

2.1. G**-actions.** Suppose a group G acts on A by algebra automorphisms, so we have a group homomorphism

$$\rho : G \to \mathrm{Aut}_{\mathrm{alg}}(A); \qquad g \mapsto \rho_g.$$

We see that A is an algebra object in the tensor category $\mathsf{Rep}(G)$, and make the following definition:

Definition 2.4. The category of G-equivariant A-modules is defined as the category of modules for A in $\mathsf{Rep}(G)$. Notation: $A\text{-mod}_{\mathsf{Rep}(G)}$.

Definition 2.5. A G-equivariant A-ring is an A-ring R equipped with an action of G by algebra automorphisms in a way compatible with the action of G on A. More precisely, we have a group homomorphism

$$\bar{\rho} : G \to \mathrm{Aut}_{\mathrm{alg}}(R); \qquad g \mapsto \bar{\rho}_g$$

such that $\bar{\rho}_g(\tau_a) = \tau_{\rho_g(a)}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} G \times A & \xrightarrow{\rho} & A \\ \downarrow{\scriptstyle 1\times\tau} & & \downarrow{\scriptstyle \tau} \\ G \times R & \xrightarrow{\bar{\rho}} & R \end{array}$$

Lemma 2.6. *If R is a G-equivariant A-ring, then R defines a monad on $A\text{-mod}_{\mathsf{Rep}(G)}$ via $M \mapsto R \otimes_A M$.*

Proof. We proceed in a way similar to the proof of Lemma 2.2 above. The coordinate-wise G-action on $R \otimes M$, given by $g \cdot (r \otimes m) = \bar{\rho}_g(r) \otimes (g \cdot m)$, descends to $R \otimes_A M$. Indeed, for any $a \in A$, we have:

$$g(r\tau_a \otimes m) = (\bar{\rho}_g(r\tau_a)) \otimes (g \cdot m) = \bar{\rho}_g(r)\tau_{\rho_g(a)} \otimes (g \cdot m), \quad \text{and}$$

$$g(r \otimes am) = \bar{\rho}_g(r) \otimes (g \cdot am) = \bar{\rho}_g(r) \otimes (\rho_g(a) \cdot (g \cdot m)).$$

[Another way to say this is that both maps $R \otimes A \otimes M \to R \otimes M$ are G-equivariant.] A simple computation shows that the A-action map $A \otimes (R \otimes_A M) \to R \otimes_A M$ is G-equivariant. The multiplication and unit of the monad are defined in the same way as in Lemma 2.2. □

Remark 2.7. The category of modules for the monad on $A\text{-mod}_{\mathsf{Rep}(G)}$ defined by R is equivalent to $R\text{-mod}_{\mathsf{Rep}(G)}$.

Remark 2.8. The operation $\otimes_A$ defines a tensor product on $A\text{-bimod}_{\mathsf{Rep}(G)}$ and R is an algebra object in this category.

2.2. Endomorphisms.

We turn our attention to the ring $(\mathrm{End}_{\mathbb{C}}(A), \circ)$ of $\mathbb{C}$-linear endomorphisms of A with

$$\tau : A \to \mathrm{End}_{\mathbb{C}}(A); \qquad a \mapsto [\tau_a : b \mapsto ab].$$

For $g \in G$, define an endomorphism $\bar{\rho}_g$ of $\mathrm{End}_{\mathbb{C}}(A)$ by $\bar{\rho}_g(f) = \rho_g \circ f \circ \rho_{g^{-1}}$.

Lemma 2.9. *The A-ring $\mathrm{End}_{\mathbb{C}}(A)$ is G-equivariant, as is its A-subring $\mathrm{Diff}(A)$ of differential operators on A.*

Proof. The morphisms $\bar{\rho}_g$ define an action of G on $\mathrm{End}_{\mathbb{C}}(A)$, and it is by algebra automorphisms since

$$\bar{\rho}_g(f) \circ \bar{\rho}_g(f') = \rho_g \circ f \circ \rho_{g^{-1}} \circ \rho_g \circ f' \circ \rho_{g^{-1}} = \rho_g \circ f \circ f' \circ \rho_{g^{-1}} = \bar{\rho}_g(f \circ f')$$

for any $f, f' \in \mathrm{End}_{\mathbb{C}}(A)$ and any $g \in G$. A simple computation shows that the following diagram commutes:

$$\begin{array}{ccc} G \times A & \xrightarrow{\rho} & A \\ \downarrow{\scriptstyle 1\times\tau} & & \downarrow{\scriptstyle \tau} \\ G \times \mathrm{End}_{\mathbb{C}}(A) & \xrightarrow{\bar{\rho}} & \mathrm{End}_{\mathbb{C}}(A) \end{array} \quad .$$

Hence $\bar{\rho}_g(\tau_a) = \tau_{\rho_g(a)}$ for any $a \in A$ and $g \in G$. If ∂ is a derivation of A, then $\bar{\rho}_g(\partial)$ is also a derivation:

$$\bar{\rho}_g(\partial)(ab) = \rho_g(\partial(\rho_{g^{-1}}(a)\rho_{g^{-1}}(b)))$$
$$= \rho_g(\partial(\rho_{g^{-1}}(a))\rho_{g^{-1}}(b) + \rho_{g^{-1}}(a)\partial(\rho_{g^{-1}}(b))) = \bar{\rho}_g(\partial)(a)b + a\bar{\rho}_g(\partial)(b).$$

The subalgebra $\mathrm{Diff}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$ of differential operators is generated as an algebra by the image of τ and by the derivations of A. Since these are preserved by the action of G, we see that G acts on $\mathrm{Diff}(A)$ by $g \cdot \theta = \rho_g \circ \theta \circ \rho_{g^{-1}}$. □

2.3. Sheaves.

Let X be a scheme with an action of an algebraic group G. By reducing to the affine case, one can show that the following sheaves are G-equivariant:

- The structure sheaf $\mathcal{O}_X$.
- The sheaf $\mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{O}_X)$ of endomorphisms of $\mathcal{O}_X$.
- The sheaf $\mathcal{D}_X$ of differential operators on X.

The last two are (in general) not algebra objects in $\mathsf{QCoh}_G(X)$, but are algebra objects in $\mathsf{QCoh}_G(X \times X)$ hence define monads on $\mathsf{QCoh}_G(X)$.

Definition 2.10. The category of weakly G-equivariant D-modules on X is defined as the category of modules for the monad $\mathcal{D}_X$ on $\mathsf{QCoh}_G(X)$.

In the affine case, this category is the same as modules for the algebra object $\Gamma(X, \mathcal{D}_X)$ in $\mathsf{Rep}(G)$.

2.4. **A note on moment maps.** Suppose now that G is a linear algebraic group. The moment map is the differential of $G \to \mathrm{Aut}_{\mathrm{alg}}(A)$, i.e. a map $\mu : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(A)$ whose image lies in the subspace of derivations of A. On the other hand, differentiating the action map $G \times \mathrm{End}_{\mathbb{C}}(A) \to \mathrm{End}_{\mathbb{C}}(A)$; $(g, f) \mapsto \rho_g \circ f \circ \rho_{g^{-1}}$ gives an action of $\mathfrak{g}$ on $\mathrm{End}_{\mathbb{C}}(A)$, denoted $\xi \triangleright f$. For any $f \in \mathrm{End}_{\mathbb{C}}(A)$ and any $\xi \in \mathfrak{g}$, one computes that:

$$\mu(\xi) \circ f - f \circ \mu(\xi) = \xi \triangleright f.$$

3. Equivariant $\mathcal{D}$-modules

3.1. **Moment maps.** Suppose a reductive group G acts on X. For $x \in X$, define $a_x : G \to X$ by $a_x(g) = g \cdot x$. It has differential $d(a_x)_e : \mathfrak{g} \to T_x X$. The image $\bar{u}_x := d(a_x)_e(u)$ of $u \in \mathfrak{g}$ is called the infinitesimal action of $u \in \mathfrak{g}$ on X at x. We obtain a map

$$\mathfrak{g} \to \Gamma(X, \Theta_X); \qquad u \mapsto \bar{u}$$

from $\mathfrak{g}$ to the space of vector fields on X, where Θ_X denotes the tangent sheaf on X. This map extends to an algebra homomorphism

$$\mu : \mathcal{U}\mathfrak{g} \to \Gamma(X, \mathcal{D}_X)$$

from the universal enveloping algebra of $\mathfrak{g}$ to the global sections of the sheaf of differential operators $\mathcal{D}_X$ on X. There is a symplectic action of G on T^*X, and there is a moment map given by:

$$T^*X \to \mathfrak{g}^*; \qquad (x, \alpha_x) \mapsto [u \mapsto \langle \alpha, \bar{u}_x \rangle].$$

We consider the following action of G on T^*X:

$$g \triangleright (x, \alpha_x) = \left(gx, d[x \mapsto g^{-1}x]^*_{gx}(\alpha_x) \right).$$

Here $d[x \mapsto g^{-1}x]_{gx}$ denotes the differential of the map $X \to X$, $x \mapsto g^{-1}x$ evaluated at the point gx.

Lemma 3.1. *The map μ is G-equivariant, where G acts on $\mathfrak{g}^*$ via the coadjoint action.*

Proof. Up to fixing conventions and being careful about g versus g^{-1}, the computation is:

$$\begin{aligned}
\mu(g \triangleright (x, \alpha_x)) &= \mu\left(gx, d[x \mapsto g^{-1}x]^*_{gx}(\alpha_x)\right) \\
&= [u \mapsto \langle d[x \mapsto g^{-1}x]^*_{gx}(\alpha_x), d[h \mapsto hgx]_e(u)\rangle] \\
&= [u \mapsto \langle \alpha_x, d[x \mapsto g^{-1}x]_{gx} \circ d[h \mapsto hgx]_e(u)\rangle] \\
&= [u \mapsto \langle \alpha_x d[h \mapsto g^{-1}hgx]_e(u)\rangle] \\
&= g \cdot [u \mapsto \langle \alpha_x, d[h \mapsto hx]_e(u)\rangle] \\
&= g \cdot \mu(x, \alpha_x)
\end{aligned}$$

□

Example 3.2. In the case of $\mathbb{C}^\times$ acting on $\mathbb{C}$ by scaling, the moment map is given by

$$\mu : T^*\mathbb{C} \simeq \mathbb{C}^2 \longrightarrow \mathrm{Lie}(\mathbb{C}^\times)^* \simeq \mathbb{C}, \qquad (x, p) \mapsto xp$$

Example 3.3. In the case of SL_2 acting on $\mathbb{C}^2$ via the natural representation, the moment map is given by

$$\mu : T^*\mathbb{C}^2 \simeq \mathbb{C}^4 \longrightarrow \mathfrak{sl}_2^*$$

$$(x, y, p, q) \mapsto \begin{cases} E \mapsto yp \\ F \mapsto xq \\ H \mapsto xp - yq \end{cases}$$

Definition 3.4. The adjoint action of $u \in \mathcal{U}(\mathfrak{g})$ on $a \in \Gamma(X, \mathcal{D}_X)$ is given by

$$u \triangleright a = \mu(u_{(1)}) \cdot a \cdot \mu(S(u_{(2)})).$$

Lemma 3.5. *The multiplication on $\Gamma(X, \mathcal{D}_X)$ is $\mathcal{U}(\mathfrak{g})$-linear, and hence $\Gamma(X, \mathcal{D}_X)$ defines an algebra object in the tensor category $\mathcal{U}(\mathfrak{g})$-*mod.

3.2. **Case of a free action.** Suppose G acts on X freely. Let $\pi : X \to X/G$ be the quotient map.

Lemma 3.6. *The cotangent bundle $T^*(X/G)$ is given by the quotient $\mu^{-1}(0)/G$.*

Proof. Fix $x \in X$ and let $\mathrm{Orb}_x \subseteq X$ be the G-orbit in X through x. Since the action of G is free, we can identify (Orb_x, x) with (G, e) as pointed spaces, and $T_x\mathrm{Orb}_x$ with $T_eG = \mathfrak{g}$. Consider the following diagrams:

$$\begin{array}{ccc} \mathrm{Orb}_x & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{\pi(x)\} & \longrightarrow & X/G \end{array} \qquad \begin{array}{ccc} T_x(\mathrm{Orb}_x) \simeq \mathfrak{g} & \longrightarrow & T_xX \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & T_{\pi(x)}(X/G) \end{array} \qquad \begin{array}{ccc} T^*_xX & \longrightarrow & \mathfrak{g}^* \\ \uparrow & & \uparrow \\ T^*_{\pi(x)}(X/G) & \longrightarrow & 0 \end{array}$$

We see that $\mu^{-1}(0)$ is a vector bundle over X whose fiber over a point $x \in X$ is $T^*_{\pi(x)}(X/G)$. Therefore, $\pi^*(T^*(X/G)) = \mu^{-1}(0)$. □

3.3. **Weakly equivariant D-modules.** Suppose G acts on X. What is the appropriate notion of a G-equivariant D-module on X? There are two kinds of equivariance for D-modules on X, weak and strong.

> **Slogan**: a weakly G-equivariant $\mathcal{D}$-module on X is a $\mathcal{D}_X$-module that is equivariant as an $\mathcal{O}_X$-module.

The category of weakly equivariant $\mathcal{D}$-modules, denoted $D_G(X)$ or $D(X/^wG)$, can be described in several ways:

- If X is affine (or just D-affine), then $D_G(X)$ is the category of $\Gamma(X, \mathcal{D}_X)$-modules in the category of $\Gamma(G, \mathcal{O}_G)$-comodules.
- The category $\mathsf{QCoh}(G)$ of quasicoherent sheaves on G is a monoidal category under convolution. The category $\mathcal{D}(X)$ of $\mathcal{D}$-modules on X is a module category $\mathcal{D}(G)$. The category of weakly equivariant $\mathcal{D}$-modules is defined as the category of $\mathsf{QCoh}(G)$-equivariant functors from the category Vect of vector spaces to $\mathcal{D}(X)$:
$$\mathcal{D}_G(X) = \mathrm{Hom}_{\mathrm{QC}(G)}(\mathsf{Vect}, \mathcal{D}(X))$$
- The category $D_G(X)$ has objects equivariant sheaf $\mathcal{F}$ equipped with a $\mathcal{D}_X$-module structure such that the isomorphism $\Phi : a^*\mathcal{F} \to p^*\mathcal{F}$ is a morphism of $\mathcal{O}_G \boxtimes \mathcal{D}_X$-modules.
- Recall that the category of D-modules on X is equivalent to the category of quasicoherent sheaves on the de Rham space X_{dR} of X, i.e. $\mathcal{D}(X) = \mathrm{QC}(X_{\mathrm{dR}})$. Then $\mathcal{D}_G(X)$ can be defined as the category of quasicoherent sheaves on the stack quotient X_{dR}/G:
$$D_G(X) = \mathrm{QC}(X_{\mathrm{dR}}/G).$$

3.4. **Strongly equivariant D-modules.**

Definition 3.7. The category $D(X/G)$ of strongly G-equivariant D-modules on X has objects given by an equivariant sheaf $\mathcal{F}$ equipped with a $\mathcal{D}_X$-module structure such that the isomorphism $\Phi : a^*\mathcal{F} \to p^*\mathcal{F}$ is a morphism of $\mathcal{D}_G \boxtimes \mathcal{D}_X$-modules.

Recall that $G_{\mathrm{dR}} = G/\hat{G}$, where $\hat{G}$ is the formal group, and $X_{\mathrm{dR}}/G_{\mathrm{dR}} = (X/G)_{\mathrm{dR}}$. Hence, the category of strongly equivariant D-modules on X is
$$\mathcal{D}(X/G) = \mathrm{QC}(X_{dR}/G_{dR}).$$
For X_{dR}/G, the $\hat{G}$-action is trivialized twice.

3.5. **Quantum Hamiltonian reduction.** We have the following Cartesian squares:

$$\begin{array}{ccc} \mu^{-1}(0) & \longrightarrow & T^*X \\ \downarrow & & \downarrow \mu \\ 0 & \longrightarrow & \mathfrak{g}^* \end{array} \qquad\qquad \begin{array}{ccc} T^*(X/G) & \longrightarrow & (T^*X)/G \\ \downarrow & & \downarrow \mu \\ 0/G & \longrightarrow & \mathfrak{g}^*/G \end{array}$$

In other words, the square on the right indicates that $T^*(X/G) = \mu^{-1}(0)/G$. The square on the right receives a map from the one on the left, obtained by quotienting by G. Consequently, we have that

$$\mathsf{QCoh}(\mu^{-1}(0)) = \mathsf{QCoh}(T^*X) \otimes_{\mathsf{QCoh}(\mathfrak{g}^*)} \mathsf{Vect}$$
$$\mathsf{QCoh}(T^*(X/G)) = \mathsf{QCoh}(T^*X/G) \otimes_{\mathsf{QCoh}(\mathfrak{g}^*/G)} \mathsf{Rep}(G).$$

We now consider deformation quantization (in the sense of replacing functions on the cotangent bundle with differential operators) of the second identity. To this end, we introduce the category of Harish-Chandra bimodules.

Definition 3.8. The category of Harish Chandra bimodules HC has several descriptions:

- $\mathcal{D}$-modules on G that are weakly equivariant for the action of $G \times G$ by left and right multiplication. So
$$\mathrm{HC} = \mathcal{D}(G\backslash^w G/^w G) = D_G\text{-mod}_{\mathcal{O}_G \otimes \mathcal{O}_G\text{-comod}}.$$
- $\mathcal{U}\mathfrak{g}$-modules that are weakly G-equivariant:
$$\mathrm{HC} = \mathcal{U}\mathfrak{g}\text{-mod}_{\mathcal{O}_G\text{-comod}}$$
- $\mathcal{U}\mathfrak{g}$-bimodules that strongly equivariant for G_Δ, i.e. that are integrable for the diagonal action of G.

From the first description, it is clear that the category HC is monoidal; it is also a deformation quantization of the category $\mathsf{QCoh}(\mathfrak{g}^*/G)$. We regard the category $D(X/^w G)$ as a right module category for HC and $\mathsf{Rep}(G) = \mathcal{D}(\bullet/^w G) = \mathcal{D}(\bullet/G)$ as a left module category for HC.

Proposition 3.9. *The category of strongly G-equivariant $\mathcal{D}$-modules on X is equivalent to the tensor product of the category of weakly G-equivariant $\mathcal{D}$-modules on X with the category* $\mathsf{Rep}(G)$ *over* HC*.*

$$\mathcal{D}(X/G) = \mathcal{D}(X/^w G) \otimes_{HC} \mathsf{Rep}(G).$$

A consequence of Gaitsgory's 1-affineness theorem is that the categories $(\mathsf{Rep}(G), \otimes)$ and $(\mathsf{QCoh}(G), *)$ are Morita equivalent. Similarly, the categories of bimodules for $\mathsf{Rep}(G)$ and $\mathsf{QCoh}(G)$ are equivalent. Under this equivalence, the algebra object $\mathsf{QCoh}(G_{\mathrm{dR}})$ corresponds to the algebra object HC. There is also the following picture. Let $\pi : X \to X/G$ be the quotient. The quantum Hamiltonian reduction of a weakly equivariant $\mathcal{D}$-module M on X is $\pi_*(M)^G$.

3.6. **Quantum group version.** We remark on the quantum version, following [BBJ18a, BBJ18b]. We replace $\mathsf{Rep}(G)$ by $\mathsf{Rep}_q(G) = U_q(\mathfrak{g})$-mod and HC by

$$\mathrm{HC}_q = \mathcal{O}_q(G)\text{-mod}_{\mathsf{Rep}_q(G)} = HH_*(\mathsf{Rep}_q(G)\text{-mod}).$$

In the TFT interpretation, $Z(\mathrm{pt}) = Z(D^2) = \mathsf{Rep}_q(G)$, which is an object in $Z(S^1) = \mathrm{HC}_q$-mod. Note that the monoidal structure on HC_q-mod is coming from stacking cylinders, not the pair of pants. So HC_q-modules are the same as braided $\mathsf{Rep}_q(G)$-modules.

Moreover,

$$\mathrm{HC}_q\text{-mod} = \int_{S^1\times\mathbb{R}} \mathsf{Rep}_q(G)\text{-mod}$$

and is also the universal enveloping algebra of the E_2-algebra $\mathsf{Rep}_q(G)$ (in categories). If $\mathcal{C}$ is a monoidal category for HC_q, then the category $\mathcal{C} \otimes_{\mathrm{HC}_q} \mathsf{Rep}_q(G)$ is the Hamiltonian reduction of $\mathcal{C}$.

Example 3.10. If S is a surface and $x \in S$ is a point on S, then the category

$$\mathsf{QCoh}_q(\mathsf{Loc}_G(S \setminus \{x\})) = \int_{S\setminus\{x\}} \mathsf{Rep}_q(G)$$

is a (right) module for HC_q, and the Hamiltonian reduction is

$$\mathcal{C} \otimes_{\mathrm{HC}_q} \mathsf{Rep}_q(G) = \int_{S\setminus\{x\}} \mathsf{Rep}_q(G) \otimes_{\int_{S\times\mathbb{R}} \mathsf{Rep}_q(G)} \int_{D^2} \mathsf{Rep}_q(G) = \int_S \mathsf{Rep}_q(G).$$

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