Notes on D-modules on $\mathbb{C}^*$

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Definition 0.1. We define the following algebras:

(1) Let $\mathcal{O}(\mathbb{C}^*) = \mathbb{C}[x^{\pm 1}]$ be the algebra of polynomial functions on $\mathbb{C}^*$.
(2) Let $D(\mathbb{C}^*)$ be the algebra of differential operators on $\mathbb{C}^*$, generated by variables $x^{\pm 1}$ and ∂ subject to the relation $[\partial, x] = 1$.
(3) Let $\mathcal{O}(\mathbb{C}) = \mathbb{C}[t]$ be the algebra of polynomial functions on $\mathbb{C}$.
(4) Let A be the algebra of difference operators on $\mathbb{C}$, generated by variables t and $\Delta^{\pm 1}$ subject to the relation $\Delta t = (t+1)\Delta$.

The algebra A acts on $\mathcal{O}(\mathbb{C}) = \mathbb{C}[t]$, where Δ takes the polynomial $f(t) \in \mathcal{O}(\mathbb{C}) = \mathbb{C}[t]$ to the polynomial $f(t+1)$. Observe that A is the smash product of the algebra $\mathcal{O}(\mathbb{C})$ with the group algebra $\mathbb{C}[\Delta^{\pm 1}] = \mathbb{C}[\mathbb{Z}]$ of the integers, where $\mathbb{Z}$ acts on $\mathbb{C}$ by translations, so $\Delta t \Delta^{-1} = t + 1$. From this, one sees that an A-module is the same as a $\mathbb{Z}$-equivariant quasicoherent sheaf on $\mathbb{C}$. We denote the category of such sheaves by $\mathsf{QCoh}^{\mathbb{Z}}(\mathbb{C})$.

Lemma 0.2. *There is an isomorphism $D \xrightarrow{\sim} A$ given by $x \mapsto \Delta^{-1}$ and $\partial \mapsto t\Delta$. Consequently there is an equivalence of categories*

$$D(\mathbb{C}^*)\text{-mod} \simeq \mathsf{QCoh}^{\mathbb{Z}}(\mathbb{C}).$$

The action of $\mathbb{C}^*$ on itself by multiplication induces an action of $\mathbb{C}^*$ on the algebra $D(\mathbb{C}^*)$, where $\lambda \in \mathbb{C}^*$ acts by $x \mapsto \lambda x$ and $\partial \mapsto \lambda^{-1}\partial$. We see that:

Lemma 0.3. *The subalgebra of $\mathbb{C}^*$-invariants of $D(\mathbb{C}^*)$ is the space of polynomials in $x\partial$.*

Definition 0.4. The category $D(\mathbb{C}^*)\text{-mod}_{\mathbb{C}^*}$ of weakly $\mathbb{C}^*$-equivariant D-modules on $\mathbb{C}^*$ is the category of modules for the subalgebra of $\mathbb{C}^*$-invariants of $D(\mathbb{C}^*)$.

Thus, weakly $\mathbb{C}^*$-equivariant D-modules on $\mathbb{C}^*$ can be identified with with quasi-coherent sheaves on $\mathbb{C}$, and we have an equivalence of categories: $D(\mathbb{C}^*)\text{-mod}_{\mathbb{C}^*} \simeq \mathsf{QCoh}(\mathbb{C})$. The following diagram is useful:

$$\begin{array}{ccc} \mathbb{C}[x\partial] & \xrightarrow{\sim} & \mathbb{C}[t] \\ \downarrow & & \downarrow \\ D(\mathbb{C}^*) & \xrightarrow{\sim} & A \end{array}$$

Remark 0.5. In general, weakly G-equivariant D-modules on a linear algebraic group G (for either the left or the right multiplication action) are the same as modules for the universal enveloping algebra of the Lie algebra of G.

We state a final, elementary lemma:

Lemma 0.6. *The forgetful functor $D(\mathbb{C}^*)\text{-mod}_{\mathbb{C}^*} \to D(\mathbb{C}^*)\text{-mod}$ from weakly equivariant D-modules to ordinary D-modules corresponds to the induction functor $\mathbb{C}[t]\text{-mod} \to A\text{-mod}$.*

$$\mathrm{Ind} : \mathsf{QCoh}(\mathbb{C}) = \mathbb{C}[t]\text{-mod} \to A\text{-mod} = \mathsf{QCoh}^{\mathbb{Z}}(\mathbb{C})$$

$$M \mapsto A \otimes_{\mathbb{C}[t]} M$$

So $\mathrm{Ind}(M) = \bigoplus_{n \in \mathbb{Z}} \Delta^n M$ as a vector space. Let $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ be the algebra homomorphism defined by $t \mapsto t - 1$. (This corresponds to action of shifting by $1 \in \mathbb{Z}$ on $\mathbb{C}$.) Define

$$\phi : \sigma^* \mathrm{Ind}(M) \to \mathrm{Ind}(M)$$
$$\Delta^n m \mapsto \Delta^{n-1} m$$

This is an isomorphism of $\mathbb{C}[t]$-modules:

$\phi(t \cdot (\Delta^n m)) = \phi(\Delta^n((t - n + 1) \cdot m)) = \Delta^{n-1}((t - n + 1) \cdot m) = t \cdot (\Delta^{n-1} m) = t \cdot \phi(\Delta^n m).$