NOTES ON FLATNESS AND THE REAL LOG CANONICAL THRESHOLD

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CONTENTS

1. INTRODUCTION

These expository notes provide an approach to motivating the definition of the real log canonical threshold. Specifically, we first define the flatness of a singularity of a real analytic variety, and then set the real log canonical threshold to be the inverse of the flatness. While the real log canonical threshold is well-established in the algebraic geometry and singular learning theory literature, the author finds that flatness is more intuitive and easier to motivate from elementary principles. The main references are [Wat09, Chapters 2 and 3] and [Car23]. We assume some familiarity with algebraic and differential geometry, and with analytic functions.

1.1. **Flatness of monomials.** We begin with the elementary geometric observation that, if $k < \ell$ are positive integers, then the graph of the function x^ℓ is "flatter" at the origin than that of x^k. Formally, this means that the preimage of any neighborhood of the origin under x^k is contained in its preimage under x^ℓ. We set the flatness of a x^k at the origin to be k:

$$\text{Flatness}\left(x^k, 0\right) := k$$

Indeed, as k increases, the flatness increases[1]. Similarly, for a monomial $x_1^{k_1} \cdots x_d^{k_d}$, the flattest direction away from the singularity at the origin is determined by the maximum exponent. We set:

$$\text{Flatness}\left(x_1^{k_1} \cdots x_d^{k_d}, 0\right) := \max_i \left(k_i\right)$$

[1]This definition makes sense for functions of the form x^r for r a nonzero real number; however, x^r is not analytic at the origin unless r is a positive integer.

What about functions that are not monomials? The functions that are closest to being monomials are the ones with normal crossings at the origin. Recall that an analytic function f defined on an open neighborhood of $\mathbb{R}^d$ is said to have *normal crossings* at the origin if, on a suitably small neighborhood of the origin, f can be written of the form[2]:

$$f(x) = a(x)x_1^{k_1} \cdots x_d^{k_d}$$

for an analytic function a with $a(0) \neq 0$ and nonnegative integers $k_1, \ldots, k_d$. In this case, we set the flatness of f at the origin to be the flatness of the monomial part:

$$\text{Flatness}(f, 0) := \text{Flatness}\left(x_1^{k_1} \cdots x_d^{k_d}, 0\right) = \max_i (k_i)$$

If $d = 1$, any real analytic function of one variable has normal crossings at the origin [KP02, Section 1.2]. However, not every function of $d > 1$ variables has normal crossings (consider the cuspidal cubic $f(x, y) = y^2 + x^3$).

1.2. **Outline.** A primary goal of these notes is to illustrate how one defines and computes the flatness of functions that do not have normal crossings. The definition relies on a deep theorem of Hironaka on resolutions of singularities, which we discuss in Section 3. On the other hand, actually computing flatness usually involves performing blowups; we provide background on blowups in Section 2. Blowups have the property that they preserve normal crossings; the precise interaction provides motivation for the general definition of flatness (which is why the section on blowups precedes the section on resolutions). We postpone a detailed discussion of examples until Section 4. We present a different, equivalent, definition of flatness in Section 5; this definition amounts to taking the leading exponent of the volume of the preimage $f^{-1}([-\epsilon, \epsilon])$ as a function of ϵ. Finally, in Appendix A, we include a general discussion on the geometry of successive blowups.

1.3. **Flatness in general.** To give a sense of the general definition of flatness, the Hironaka's theorem asserts that, given a real analytic function f defined on an open neighborhood of the origin in $\mathbb{R}^d$, there is (roughly speaking) a blowup map $g : \mathcal{M} \to \mathbb{R}^d$ such the pullback of f to any coordinate chart has normal crossings at any point in the fiber over the origin. Then the flatness of f is defined as the maximum of the flatness of its normal crossings pullbacks, divided by a cost coming from the determinant of the Jacobian of the blowup map g.

The map g is not unique, due in part to the fact that the pullback of a function with normal crossings under a blowup again has normal crossings. Therefore, the cost is defined in a way not to depend on the particular choice of blowup. In particular, if f happens to be a monomial, then its flatness computed directly (i.e. the maximum exponent) will match the flatness computed via any blowup.

[2]This definition usually includes the assumption that $f(0) = 0$; this does not seem necessary for our purposes. If $f(0) \neq 0$, then one can take $k_i = 0$ for all i, and the flatness is 0.

We remark that the approach outlined above defines the flatness at the origin; for any other point $x^* \in \mathbb{R}^d$, we set:

$$\text{Flatness}\,(f, x^*) := \text{Flatness}\,(x \mapsto f(x + x^*), 0)$$

Moreover, the global flatness of f is the maximum flatness over all points in the domain of f:

$$\text{Flatness}(f) := \max_{x^* \in \text{dom}(f)} \text{Flatness}\,(f, x^*)$$

In fact, since the flatness is zero at points where $f(x^*) \neq 0$, we can take the maximum only over $f^{-1}(0)$, as long as this set is nonempty. Finally, the real log canonical threshold (local or global) is the inverse of the flatness:

$$\text{RLCT}(f, p) := \frac{1}{\text{Flatness}(f, p)} \qquad \text{RLCT}(f) := \min_{p \in f^{-1}(0)} \text{RLCT}(f, p)$$

If the flatness at a point is zero, then the real log canonical threshold is set to be $+\infty$ at that point. We now turn our attention to blowups, before returning to flatness and the real log canonical threshold in Section 3.

2. Blowups

In this section, we define blowups precisely, and collect results which will be relevant in later sections. Recall that projective space $\mathbb{P}^{d-1}$ is the moduli space of lines through the origin in $\mathbb{R}^d$.

2.1. **Incidence relation.**

The blowup of the origin in $\mathbb{R}^d$, for $d \geq 2$, is defined as the following space via an incidence relation:

$$B_{\{0\}}(\mathbb{R}^d) = \left\{ (x, \ell) \in \mathbb{R}^d \times \mathbb{P}^{d-1} \mid x \in \ell \right\}$$

In other words, we consider the set of all pairs (x, ℓ) of a point x in $\mathbb{R}^d$ and a line ℓ through the origin in $\mathbb{R}^d$ such that the point lies on the line. We have the two projections:

$$\begin{array}{ccccc} & & B_{\{0\}}(\mathbb{R}^d) & & \\ & \overset{\mu}{\swarrow} & & \overset{p}{\searrow} & \\ \mathbb{R}^d & & & & \mathbb{P}^{d-1} \end{array}$$

The fiber of μ of any nonzero $x \in \mathbb{R}^d$ is a single point since there is a unique line passing through the origin and x. Meanwhile, the fiber over $0 \in \mathbb{R}^d$ is a copy of $\mathbb{P}^{d-1}$, since the latter parametrizes lines through the origin. Thus, as a family over $\mathbb{R}^d$, the blow-up is a bijection over the complement of the origin and a copy of projective space over the origin:

$$\begin{array}{ccccc}
\mathbb{P}^{d-1} & \hookrightarrow & B_{\{0\}}(\mathbb{R}^d) & \hookleftarrow & \mathbb{R}^d \setminus \{0\} \\
\downarrow & & \downarrow \mu & & \downarrow \simeq \\
\{0\} & \hookrightarrow & \mathbb{R}^d & \hookleftarrow & \mathbb{R}^d \setminus \{0\}
\end{array}$$

On the other hand, the projection $p : B_{\{0\}}(\mathbb{R}^d) \to \mathbb{P}^{d-1}$ is a fibration with fiber $\mathbb{R}$. Indeed, the fiber over any line is the set of points on that line, so the blow-up is the total space of the tautological bundle over $\mathbb{P}^{d-1}$. In the case $d = 2$, this total space, and hence the blow up itself, is topologically a Möbius strip. The zero section is the fiber $\mu^{-1}(0)$. This description shows that the blow up can be given the structure of either a smooth manifold, a real analytic manifold, or a smooth real algebraic variety. For our purposes, we mostly focus on the real analytic structure, but comment on the other ones where relevant.

2.2. **Equations.** To give another description of the blow up, we first note that two points $x, y \in \mathbb{R}^d$ lie on the same line through the origin if and only if $x_i y_j = x_j y_i$ for all $1 \leq i, j \leq d$. (The proof of this fact is elementary.) Hence, using homogenous coordinates on $\mathbb{P}^{d-1}$, we obtain:

$$B_{\{0\}}(\mathbb{R}^d) = \left\{ (x, \ell) \in \mathbb{R}^d \times \mathbb{P}^{d-1} \mid x_i \ell_j = x_j \ell_i \text{ for } 1 \leq i, j \leq d \right\}$$

For example, in the case $d = 2$, we have:

$$B_{\{0\}}(\mathbb{R}^2) = \left\{ (x, \ell) \in \mathbb{R}^2 \times \mathbb{P}^1 \mid \det \begin{bmatrix} x_1 & x_2 \\ \ell_1 & \ell_2 \end{bmatrix} = 0 \right\}$$

2.3. **Closure.** We illustrate a realization of the blow up as a closure. Consider the map

$$a : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d \times \mathbb{P}^{d-1}, \qquad x \mapsto (x, [x])$$

In other words, we include into the first factor and project to projective space in the second factor. The image consists of pairs (x, ℓ) of a point in $\mathbb{R}^d \setminus \{0\}$ and the line in $\mathbb{R}^d$ containing that point.

Lemma 2.1. *The Euclidean closure of the image of a is precisely the blow-up $B_{\{0\}}(\mathbb{R}^2)$.*

Sketch of proof. The inclusion of the closure in $B_{\{0\}}(\mathbb{R}^d)$ is immediate. For the opposite inclusion, we show that $(0, \ell)$ belongs to the closure of the image of a for any $\ell = [\ell_1 : \cdots : \ell_d] \in \mathbb{P}^{d-1}$. Indeed, the image under a of the sequence $n \mapsto \left(\frac{\ell_1}{n}, \ldots, \frac{\ell_d}{n} \right)$ in $\mathbb{R}^d \setminus \{0\}$, for $n = 1, 2, \ldots,$ limits to $(0, \ell)$. □

We have used the Euclidean topology when speaking of the closure. However, a is map of varieties, and the closure of its image in the Zariski topology is also $B_{\{0\}}(\mathbb{R}^2)$. This follows from the following facts: (1) the Euclidean topology is finer than the Zariski topology, so the Zariski closure contains the Euclidean closure and (2) $B_{\{0\}}(\mathbb{R}^2)$ is closed in the Zariski topology which means the former is contained in the Zariski closure.

2.4. **Coordinate charts.** To perform calculations on the blow up, it will be convenient to introduce a collection of coordinate charts. For $i = 1, \dots, d$, recall the canonical open sets of $\mathbb{P}^{d-1}$ given by:

$$\mathbb{P}^{d-1}[i] = \left\{ \ell = [\ell_1 : \cdots : \ell_2] \in \mathbb{P}^{d-1} \mid \ell_i \neq 0 \right\}$$

Set U_i to be the preimage of $\mathbb{P}^{d-1}[i]$ under the map p:

$$\begin{aligned} U_i &= \left\{ (x, \ell) \in \mathbb{R}^d \times \mathbb{P}^{d-1} \mid \ell_i \neq 0 \ , \ x_j \ell_k = x_k \ell_j \text{ for } j, k = 1, \dots d \right\} \\ &= \left\{ (x, \ell) \in \mathbb{R}^d \times \mathbb{P}^{d-1} \mid \ell_i \neq 0 \ , \ x_j = x_i \frac{\ell_j}{\ell_i} \text{ for } j = 1, \dots d \right\} \end{aligned}$$

We see that there are coordinate charts:

$$\psi_i : \mathbb{R}^d \to U_i, \qquad (\mu \circ \psi_i(z))_j = \begin{cases} z_i & \text{if } i = j \\ z_i z_j & \text{if } i \neq j \end{cases}$$
$$p \circ \psi_i(z) = [z_1 : \cdots : z_{i-1} : 1 : z_{i+1} : \cdots : z_d]$$

where we use the fact that a map to U_i is the same as a pair of maps, one to $\mathbb{R}^d$ and another to $\mathbb{P}^{d-1}[i]$, such that the incidence condition is satisfied, which is the case: $\mu \circ \psi_i(z) \in p \circ \psi_i(z)$. The inverse of ψ_i is given by:

$$\phi_i : U_i \to \mathbb{R}^d, \qquad \phi_i(x, \ell)_j = \begin{cases} x_i & \text{if } i = j \\ \frac{\ell_j}{\ell_i} & \text{if } i \neq j \end{cases}$$

To write down the transition functions, first observe that the image of the intersection $U_i \cap U_j$ under ϕ_i is given by the complement of the z_j coordinate hyperplane:

$$\phi_i(U_i \cap U_j) = \{ z \in \mathbb{R}^d \mid z_j \neq 0 \}$$

The transition functions are given by:

$$\alpha_{ij} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j), \qquad \alpha_{ij}(z)_k = \begin{cases} \frac{1}{z_j} & \text{if } k = i \\ z_i z_j & \text{if } k = j \\ \frac{z_k}{z_j} & \text{otherwise} \end{cases}$$

We conclude that the U_i form an atlas on $B_{\{0\}}(\mathbb{R}^d)$. We will refer to the U_i as the *canonical open sets* coving the blowup $B_{\{0\}}(\mathbb{R}^d)$.

Since U_i is defined as $p^{-1}(\mathbb{P}^{d-1}[i])$, its image under p is $\mathbb{P}^{d-1}[i]$. On the other hand, the image of U_i under μ is the set of points where the coordinate x_i being zero implies that all other coordinates are zero:

$$\mu(U_i) = \{ x \in \mathbb{R}^d \mid x_i \neq 0 \} \cup \{0\}.$$

Equivalently, the image of U_i under μ is the complement of the punctured x_i-axis in $\mathbb{R}^d$.

Remark 2.2. For future reference, we note that the determinant of the Jacobian of $\mu \circ \psi_i$ at z is given by $\det(\mathrm{Jac}_z(\mu \circ \psi_i)) = z_i^{d-1}$. The verification of this claim is an exercise; one can reduce to the case $i = 1$.

Remark 2.3. Later in these notes we will consider pulling back functions from $\mathbb{R}^d$ to the coordinate charts U_i via $\mu \circ \psi_i$. For example, if $f(x) = x_1^{k_1} \cdots x_d^{k_d}$ is a monomial, then,

$$f \circ \mu \circ \psi_i(z) = z_1^{k_1} \cdots z_{i-1}^{k_{i-1}} \left(z_i^{\sum_j k_j} \right) z_{i+1}^{k_{i+1}} \cdots z_d^{k_d}$$

so that the exponent of z_i is the sum of the original exponents, while all other exponents remain unchanged.

Example 2.4. When $d = 2$, we have the open sets:

$$U_1 = \left\{ (x, \ell) \in \mathbb{R}^2 \times \mathbb{P}^1 \mid \ell_1 \neq 0 \, , \; x_2 = x_1 \frac{\ell_2}{\ell_1} \right\}$$
$$U_2 = \left\{ (x, \ell) \in \mathbb{R}^2 \times \mathbb{P}^1 \mid \ell_2 \neq 0 \, , \; x_1 = x_2 \frac{\ell_1}{\ell_2} \right\}$$

with coordinates given by:

$$\phi_1(x, \ell) = \left(x_1, \frac{\ell_2}{\ell_1} \right) \qquad \phi_2(x, \ell) = \left(\frac{\ell_1}{\ell_2}, x_2 \right)$$
$$\psi_1(z) = ((z_1, z_1 z_2) \, , \; [1 : z_2]) \qquad \psi_2(z) = ((z_1 z_2, z_2) \, , \; [z_1 : 1])$$

and transition maps:

$$\alpha_{12} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2), \qquad (z_1, z_2) \mapsto \left(\frac{1}{z_2}, z_1 z_2 \right)$$
$$\alpha_{21} : \phi_2(U_1 \cap U_2) \to \phi_1(U_1 \cap U_2), \qquad (y_1, y_2) \mapsto \left(y_1 y_2, \frac{1}{y_1} \right)$$

The Jacobian of $\mu \circ \phi_1$ at z is easily seen to be $\begin{bmatrix} 1 & 0 \\ z_2 & z_1 \end{bmatrix}$, with determinant z_1; similarly, the determinant of the Jacobian of $\mu \circ \phi_2$ is z_2. The pullbacks of a monomial $x_1^{k_1} x_2^{k_2}$ to U_1 and U_2 are, respectively, $x_1^{k_1+k_2} x_2^{k_2}$ and to $x_1^{k_1} x_2^{k_1+k_2}$.

2.5. **Blowups and normal crossings.** An important property of blowups is that they preserve normal crossings. Specifically, suppose f is a real analytic function defined[3] on $\mathbb{R}^d$, and let $\mu : B_{\{0\}}(\mathbb{R}^d) \to \mathbb{R}^d$ be the blowup at the origin. For any $i = 1, \ldots, d$, the pullback of f under $\mu \circ \psi_i$ is a function $\mathbb{R}^d \to \mathbb{R}$.

Proposition 2.5. *Let f be as above. If f has normal crossings at the origin, then, for $i = 1, \ldots, d$, the pullback of f under $\mu \circ \psi_i$ has normal crossings at any point in $(\mu \circ \psi_i)^{-1}(0)$.*

Sketch of proof. First, observe that the fiber $(\mu \circ \psi_i)^{-1}(0)$ is equal to $\{z_i = 0\} \subset \mathbb{R}^d$. Let $p = (\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_d)$ be a point in this fiber. Let $\tau_p : \mathbb{R}^d \to \mathbb{R}^d$ be translation by p, so that $\tau_p(z) = z + p$. Then:

$$\mu \circ \psi_i \circ \tau_p(z) = \begin{cases} z_i & \text{if } j = i \\ z_i(z_j + \lambda_j) & \text{if } j \neq i \end{cases}$$

[3]The results below hold when f is defined only on a neighborhood of the origin in $\mathbb{R}^d$.

The proposition follows from showing that the pullback of f under $\mu \circ \psi_i \circ \tau_p$ has normal crossings at the origin. Since f itself has normal crossings at the origin, we can write:

$$f(x) = a(x) x_1^{k_1} \cdots x_d^{k_d}$$

for non-negative integers k_j and a real analytic function a such that $a(0) \neq 0$. Therefore, the pullback of f under $\mu \circ \psi_i$, shifted by p, is given by:

$$f \circ \mu \circ \psi_i \circ \tau_p(z) = a(\mu \circ \psi_i \circ \tau_p(z)) z_i^{\sum_j k_j} \prod_{j \neq i} (z_j + \lambda_j)^{k_j}$$

Set:

$$\tilde{a}(z) = a(\mu \circ \psi_i \circ \tau_p(z)) \prod_{j : \lambda_j \neq 0} (z_j + \lambda_j)^{k_j}$$

Then $\tilde{a}(0) = a(0) \prod_{j:\lambda_j \neq 0} \lambda_j^{k_j} \neq 0$, and

$$f \circ \mu \circ \psi_i \circ \tau_p(z) = \tilde{a}(z) z_i^{\sum_j k_j} \prod_{j \neq i, \lambda_j = 0} z_j^{k_j} \tag{2.1}$$

The result follows. □

In Section 1.1, we defined the flatness of an analytic function with normal crossings. Since normal crossings are preserved by blowups, we can compare the flatness of a function with normal crossings to the flatness of its pullback under a blowup. More precisely, suppose $f(x) = a(x) x_1^{k_1} \cdots x_d^{k_d}$ has normal crossings at the origin, where the k_j are non-negative integers and $a(0) \neq 0$. Hence, the flatness of f at the origin is the maximum of the k_j. Now, fix $i \in \{1, \ldots, d\}$, and let $p = (\lambda_1, \ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots, \lambda_d)$ be a point in the fiber $(\mu \circ \psi_i)^{-1}(0)$. Let $\tau_p : \mathbb{R}^d \to \mathbb{R}^d$ be translation by p. Then we see from Equation 2.1 that the flatness of the pullback is the sum of the exponents:

$$\begin{aligned} \text{Flatness}(f \circ \mu \circ \psi_i, p) &= \text{Flatness}(f \circ \mu \circ \psi_i \circ \tau_p, 0) \\ &= \max \left(\sum_j k_j \,,\, \max_{j \neq i, \lambda_j = 0} k_j \right) = \sum_j k_j \end{aligned}$$

This mismatch can be corrected by incorporating a cost of blowing up. Specifically, instead of comparing $\sum_j k_j$ to each of the k_i (the former will always be larger), we compare the average $\frac{\sum_j k_j}{d}$ to the other exponents. We set the flatness of f at the origin relative to the coordinate chart $\mu \circ \psi_i \circ \tau_p$ of the blowup to be:

$$\text{Flatness}\left(f, 0 \mid \mu \circ \psi_i \circ \tau_p\right) := \max \left(\frac{\sum_j k_j}{d} \,,\, \max_{j \neq i, \lambda_j = 0} k_j \right) \tag{2.2}$$

The point in $(\mu \circ \psi_i)^{-1}(0)$ with maximum flatness is the origin:

$$\text{Flatness}\left(f, 0 \mid \mu \circ \psi_i\right) = \max \left(\frac{\sum_j k_j}{d} \,,\, \max_{j \neq i} k_j \right) = \begin{cases} \frac{\sum_j k_j}{d} & \text{if } \max_j k_j = k_i \\ \max_j \left(k_j\right) & \text{otherwise} \end{cases}$$

We recover the flatness of f at the origin by taking the maximum over $i = 1, \dots, d$:

$$\text{Flatness}(f, 0) = \max_i \left(\text{Flatness}(f, 0 \mid \mu \circ \psi_i,)\right) = \max_j \left(k_j\right)$$

In Section 3.2 below, we generalize this procedure beyond simple blowups.

2.6. **Blowups of higher-dimensional submanifolds.** We generalize from blowing up the origin in $\mathbb{R}^d$ to blowing up the submanifold of $\mathbb{R}^n$ given by the vanishing of d coordinates. Here $n \geq 2$ is an integer, and we set $2 \leq d \leq n$. Consider the codimension d submanifold of $\mathbb{R}^n$ given by:

$$V_d = \{0\} \times \mathbb{R}^{n-d} = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \cdots = x_d = 0\}$$

We recover the previous discussion with $d = n$. The blow up of V_d in $\mathbb{R}^n$ is defined as:

$$B_{V_d}(\mathbb{R}^n) = B_{\{0\}}(\mathbb{R}^d) \times \mathbb{R}^{n-d}$$

or, equivalently, via an incidence relation $B_{V_d}(\mathbb{R}^n) = \{(x, \ell) \in \mathbb{R}^n \times \mathbb{P}^{d-1} \mid \pi_d(x) \in \ell\}$, where $\pi_d : \mathbb{R}^n \to \mathbb{R}^d$ is the projection onto the first d coordinates and we regard $\mathbb{P}^{d-1}$ as parameterizing lines through the origin in $\mathbb{R}^d$. We have projections:

$$\begin{array}{ccc} & B_{V_d}(\mathbb{R}^n) & \\ \swarrow^{\mu} & & \searrow^{p} \\ \mathbb{R}^n & & \mathbb{P}^{d-1} \end{array}$$

The fiber of μ over $x \in \mathbb{R}^n \setminus V_d$ is a single point, while the fiber of μ over $x \in V_d$ is a copy of $\mathbb{P}^{d-1}$. Thus, we have "blown up" each point of the submanifold V_d into a copy of projective space while keeping the rest of $\mathbb{R}^n$ unchanged:

$$\begin{array}{ccccc} V_d \times \mathbb{P}^{d-1} & \hookrightarrow & B_{V_d}(\mathbb{R}^n) & \hookleftarrow & \mathbb{R}^n \setminus V_d \\ \downarrow \pi_1 & & \downarrow \mu & & \downarrow \simeq \\ V_d & \hookrightarrow & \mathbb{R}^n & \hookleftarrow & \mathbb{R}^n \setminus V_d \end{array}$$

The projection $p : B_{V_d}(\mathbb{R}^n) \to \mathbb{P}^{d-1}$ is a fibration with fiber $\mathbb{R} \times \mathbb{R}^{n-d}$; the blow up can be identified with the total space of the product of the tautological bundle and the trivial $(n-d)$-dimensional bundle. We also have the map

$$a : \mathbb{R}^n \setminus V_d \to \mathbb{R}^n \times \mathbb{P}^{d-1}, \qquad x \mapsto (x, [\pi_d(x)])$$

Since $\mathbb{R}^n \setminus V_d = (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^{n-d}$, the arguments from the case $n = d$ can be used to show that the closure of a is $B_{V_d}(\mathbb{R}^n)$.

In the above discussion, we have blown up the subvariety V_d or $\mathbb{R}^n$. However, the construction works just as well when restricting to an open subset W of the origin in $\mathbb{R}^n$ and blowing up $V_d \cap W$. We write $B_{V_d \cap W}(W)$, or simply $B_{V_d}(W)$, for this blow up.

3. Resolutions

We are now ready to define the flatness of functions that do not have normal crossings at the origin using a theorem of Hironaka.

3.1. **Hironaka theorem.** To state the theorem precisely, let $\mathcal{M}$ be a d-dimensional real analytic manifold. A coordinate chart at $p \in \mathcal{M}$ is a pair (V, ψ) where V is an open neighborhood of the origin in $\mathbb{R}^d$ and ψ is an injective real analytic map $V \hookrightarrow \mathcal{M}$ with $\psi(0) = p$. We state a simplified version of the theorem of Hironaka, similar to the version given in [Wat09].

Theorem 3.1 (Hironaka)**.** *Let f be a non-constant real analytic function defined on a nonempty open neighborhood of the origin in $\mathbb{R}^d$. Then there exists:*

- *a d-dimensional real analytic manifold $\mathcal{M}$, and*
- *a proper real analytic map $g : \mathcal{M} \to W$, where $W \subseteq \mathbb{R}^d$ is an open neighborhood of the origin contained in the domain of f,*

such that:

- *the map g restricts to an analytic isomorphism $\mathcal{M} \setminus \mathcal{M}_0 \to W \setminus \{0\}$, where $\mathcal{M}_0 = g^{-1}(0)$, and*
- *for every $p \in \mathcal{M}_0$, there is a coordinate chart (V, ψ) at p such that:*

$$f \circ g \circ \psi(v) = a(v) v_1^{k_1} \cdots v_d^{k_d}$$
$$\det\left(\mathrm{Jac}_v\left(g \circ \psi\right)\right) = b(v) v_1^{h_1} \cdots v_d^{h_d}$$

  for all $v \in V$, where a and b are real analytic functions with $a(0) \neq 0$ and $b(0) \neq 0$, and the k_i and h_i are nonnegative integers.

We note that $g \circ \psi$ is a map from the open set $V \subseteq \mathbb{R}^d$ to the open set $W \subseteq \mathbb{R}^d$, and hence its Jacobian at any $v \in V$ is a d by d matrix; we take the determinant of this matrix. This procedure gives a function $V \to \mathbb{R}$ taking v to $\det\left(\mathrm{Jac}_v\left(g \circ \psi\right)\right)$.

3.2. **Flatness in general.** We now use the theorem above to define the flatness at the origin. We adopt all notation from the theorem. Given $p \in \mathcal{M}_0$ and a coordinate chart (V, ψ) at p satisfying the conditions of the theorem, we set:

$$\mathrm{Flatness}\left(f, 0 \mid g \circ \psi\right) = \max_{1 \leq i \leq n} \left(\frac{k_i}{h_i + 1}\right) \tag{3.1}$$

In other words, the flatness of f at the origin relative to the map $g \circ \psi$ is defined as the flatness of the pullback $f \circ g \circ \psi$, with an "averaging" cost in the denominator coming from the Jacobian of $g \circ \psi$.

Remark 3.2. Equation 3.1 is a direct generalization of the case when $g = \mu$ is a blowup map and f has normal crossings, see Equation 2.2. In that case, for any p in the fiber of μ, we have $p = \psi_i(z)$ for some $i = 1, \ldots, d$ and $z \in \mathbb{R}^d$; we can take the coordinate chart at p given by $\mu \circ \psi_i \circ \tau_z$. Recall from Remark 2.2 that $\det(\mathrm{Jac}_v(\mu \circ \psi_i \circ \tau_z)) = v_i^{d-1}$, so that

$h_i = d-1$ and $h_j = 0$ for $j \neq i$. On the other hand, the i-th normal crossing exponent of the pullback of f is the sum of the normal crossing exponents of f, while the other normal crossing exponents are unchanged.

It is a fact that different coordinate charts at p (satisfying the conditions of the theorem) give the same flatness, so we select one, denoted (V, ψ), and set:

$$\text{Flatness}\,(f, 0 \mid p) = \text{Flatness}\,(f, 0 \mid g \circ \psi)$$

Next, we set the flatness of f at x to be the maximum flatness of points $p \in \mathcal{M}_0$:

$$\text{Flatness}\,(f, 0) = \max_{p \in \mathcal{M}_0} \text{Flatness}\,(f, 0 \mid p)$$

For any other point x^* in the domain of f, we define the flatness by translating the function so that the point of interest is the origin:

$$\text{Flatness}(f, x^*) := \text{Flatness}(x \mapsto f(x + x^*), 0)$$

Finally, the (global) flatness of f is the maximum of the flatness at all points in the domain of f:

$$\text{Flatness}(f) = \max_{x^* \in \text{dom}(f)} \text{Flatness}(f, x^*)$$

Since the flatness is zero when $f(x^*) \neq 0$, we can take the maximum only over $f^{-1}(0)$, as long as the latter set is nonempty.

Example 3.3. Suppose functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy $f \circ g(x) = x^k$ for a nonnegative integer k, $g'(x) = x^h$ for a nonnegative integer h, and $g(0) = 0$. Then, up to a constant, $g(x) = x^{h+1}$ so that

$$x^k = f \circ g(x) = f\left(x^{h+1}\right).$$

It follows that, up to a constant, $f(x) = x^{\frac{k}{h+1}}$, so that the flatness of f is $\frac{k}{h+1}$. A warning is in order: the function x^r is not analytic at zero unless r is a non-negative integer.

Definition 3.4. Let f be a real analytic function defined on an open neighborhood of the origin in $\mathbb{R}^d$. For any x^* in the domain of f, the real log canonical threshold is defined as the inverse of the flatness:

$$\text{RLCT}(f, x^*) = \frac{1}{\text{Flatness}(f, x^*)}$$

If the flatness is zero, the real log canonical threshold defined as $+\infty$.

4. Examples

We present a number of examples illustrating techniques for computing the flatness (or, equivalently, the real log canonical threshold).

4.1. **Tree notation.** This section can be skipped on first reading and referenced when the reader arrives at the tree diagrams below. The examples below involve pulling back functions along blowups and along diffeomorphisms, and can be summarized with a (directed, rooted) tree. In the trees below, each vertex v is a labeled with a function $K_v : \mathbb{R}^d \to \mathbb{R}$. Each directed edge corresponds to a transformation $\mathbb{R}^d \to \mathbb{R}^d$; the function at the target of the edge is the pullback of the function at the source. In symbols:

$$K_{t(e)} = K_{s(e)} \circ \phi_e$$

where $s(e)$ and $t(e)$ are the source and target of the edge e, respectively. The number of emanating edges from a vertex carries various information:

- A single emanating edge e indicates that the transformation ϕ_e is a diffeomorphism.
- A count of $n > 1$ emanating edges indicates a blow up along a subset of n coordinate axes $x_{i_1} = x_{i_2} = \cdots = x_{i_n} = 0$, where $i_1 < i_2 < \cdots < i_n$ and $i_j \in \{1, \ldots, d\}$. The j-th edge corresponds to the transformation:

$$(x_1, \ldots, x_d)_k = \begin{cases} x_{i_j} & k = i_j \\ x_{i_j} x_k & \text{if } k = i_{j'} \text{ for some } j' \in \{1, \ldots, n\} \setminus \{j\} \\ x_k & \text{otherwise} \end{cases}$$

- A leaf has no emanating edges. There is a unique path from the root to the leaf ℓ; composing the maps along the path in reverse order, we obtain a transformation ϕ_ℓ of $\mathbb{R}^d$. The function K_ℓ has normal crossings at every point in the relevant fiber of ϕ_ℓ.

The notation and conventions will become clearer by examining examples.

4.2. **Squared norm.** Consider the function $f(x, y) = x^2 + y^2$, whose vanishing set is the origin. We blow up at the origin and pull back f to the two canonical coordinate charts to obtain:

$$(f \circ \mu \circ \psi_1)(x, y) = x^2(1 + y^2) \qquad \text{and} \qquad (f \circ \mu \circ \psi_2)(x, y) = y^2(x^2 + 1)$$

As the functions $1 + x^2$ and $1 + y^2$ are non-vanishing, these pullbacks have normal crossings. The Jacobian determinants are:

$$\det\left(\mathrm{Jac}_{(x,y)}(\mu \circ \psi_1)\right) = x \qquad \text{and} \qquad \det\left(\mathrm{Jac}_{(x,y)}(\mu \circ \psi_2)\right) = y$$

We conclude that the flatness is:

$$\mathrm{Flatness}(x^2 + y^2, 0) = \max\left(\frac{2}{1+1}, \frac{2}{1+1}\right) = 1$$

The real log canonical threshold is also equal to 1. The tree diagram (as per Section 4.1) is given by:

$$x^2 + y^2$$

$$x^2(1+y^2) \qquad\qquad y^2(x^2+1)$$

In this simple example, there are two edges e_1 and e_2, and two leaves ℓ_1 and ℓ_2, enumerated from left to right. The edge e_i corresponds to the transformation $\mu \circ \psi_i$, which is the restriction of the blowup at the origin to the i-th canonical coordinate chart. In the notation from Section 4.1:

$$\phi_{\ell_1} = \phi_{e_1}(x,y) = (x, xy) \qquad \text{and} \qquad \phi_{\ell_2} = \phi_{e_2}(x,y) = (xy, y)$$

In notation similar to that of [Wat09, Example 3.16], these can be summarized as:

$$x = x_1 = x_2y_2$$
$$y = x_1y_1 = y_2$$

4.3. **Cuspidal cubic.** Consider the function $f(x,y) = x^3 + y^2$, whose vanishing set is a cuspidal cubic. There is a singularity at the origin (both partial derivatives of the irreducible polynomial f vanish there). We resolve this singularity to normal crossings by blowing up three times. After the first blow up at the origin, the pullbacks of f to U_1 and U_2 are given by:

$$x^3 + x^2y^2 \qquad \text{and} \qquad x^3y^3 + y^2 = y^2(x^3y+1)$$

We have achieved normal crossings on U_2; indeed, the fiber of the map $U_2 \to \mathbb{R}^2$ over the origin is $\{y = 0\}$, so the non-monomial factor $x^3y + 1$ is constant at 1 on the fiber. Meanwhile, we blow up U_1 at the origin. The pullbacks to U_{11} and U_{12} are given by

$$x^3 + x^4y^2 = x^3(1+xy^2) \qquad \text{and} \qquad x^3y^3 + x^2y^4$$

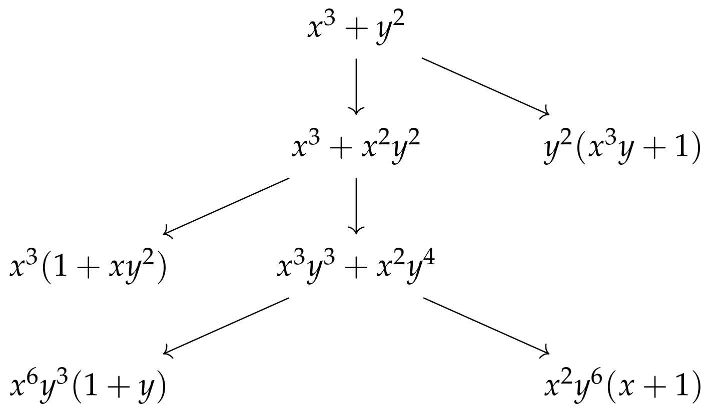
We have achieved normal crossings on U_{11}; indeed, the fiber of the map $U_{11} \to \mathbb{R}^2$ over the origin is $\{x = 0\}$, so the non-monomial factor $1 + xy^2$ is constant at 1 on the fiber. Meanwhile, we blow up U_{12} at the origin. The pullbacks to U_{121} and U_{122} are given by

$$x^6y^3 + x^6y^4 = x^6y^3(1+y) \qquad \text{and} \qquad x^3y^6 + x^2y^6 = x^2y^6(x+1).$$

These are the pullbacks of the original function f under $(x,y) \mapsto (x^2y, x^3y^2)$ and $(x,y) \mapsto (xy^2, xy^3)$, respectively. The fiber over the origin in both cases is $xy = 0$. We claim that $x^6y^3(1+y)$ has normal crossings on $\{xy = 0\}$. The only points to consider are those in the intersection of $\{xy = 0\}$ and $\{x^6y^3(1+y) = 0\}$. There are four cases:

- At $(0,0)$, we take $a(x,y) = 1 + y$.
- At $(0,-1)$, we take $a(x,y) = y^3$.
- At $(0,\lambda)$, for $\lambda \neq 0,-1$, we take $a(x,y) = y^3(1+y)$.
- At $(\lambda,0)$, for $\lambda \neq 0$, we take $a(x,y) = x^6(y+1)$.

A similar argument shows that $x^2y^6(x+1)$ has normal crossings on the fiber $\{xy = 0\}$. Thus, we have used blowups to resolve to normal crossings. We summarize the calculations with a tree:



In this example, the left arrow emanating form a vertex corresponds to the transformation $(x, y) \mapsto (x, xy)$, which is the restriction of the blowup at the origin to the first canonical coordinate chart. The right arrow corresponds to $(x, y) \mapsto (xy, y)$. There are four leaves, which we enumerate in the order they appear in a left-to-right breath first search from the root. In notation similar to that of [Wat09, Example 3.16], we summarize the maps corresponding to each leaf by:

$$\begin{aligned} x &= x_1y_1 && = x_2 && = x_3^2y_3 && = x_4y_3^2 \\ y &= y_1 && = x_2^2y_2 && = x_2^3y_3^2 && = x_4y_4^3 \end{aligned}$$

(This is shorthand for writing $\phi_1(x, y) = (xy, y)$, etc.) Computing Jacobians and factored monomials, we see that the flatness is $6/5$. Equivalently, the real log canonical threshold is $5/6$.

4.4. **Example 3.18.** Consider the function $K : \mathbb{R}^3 \to \mathbb{R}$ given by

$$K(a, b, c) = (ab + c)^2 + 3a^2b^4$$

This is a multiple of the function appearing in Example 3.18 in [Wat09]. A straightforward calculation shows that

$$K^{-1}(0) = \{(a, b, 0) \in \mathbb{R}^3 \mid ab = 0\}$$

In other words, the algebraic set defined by K is the union of the a-axis and the b-axis. Outside of this set, K takes positive values. We use blowups and diffeomorphisms to produce normal crossings at each point in $K^{-1}(0)$, proceeding first along the a-axis, and then the b-axis.

4.4.1. *Along the a-axis.* To produce normal crossings at $(\lambda, 0, 0)$, we first pull back K along the translation $(a, b, c) \mapsto (a + \lambda, b, c)$ so that the point of interest is the origin:

$$(ab + \lambda b + c)^2 + 3(a + \lambda)^2b^4$$

Next, we pull back along the invertible linear map $(a, b, c) \mapsto (a, b, c - \lambda b)$ to obtain:

$$(ab + c)^2 + 3(a + \lambda)^2 b^4$$

We then blow up along $b = c = 0$, and pull back to the two canonical coordinate charts to obtain:

$$b^2 \left((a + c)^2 + 3(a + \lambda)^2 b^2 \right) \qquad \text{and} \qquad c^2 \left((ab + 1)^2 + 3(a + \lambda)^2 b^4 c^2 \right)$$

We claim that the second of these has normal crossings on the relevant fiber. To see this, observe that the function is the pullback of K under the map $(a, b, c) \mapsto (a + \lambda, bc, c - \lambda bc)$, and the fiber of this map over $(\lambda, 0, 0)$ is $\{a = c = 0\}$. Factoring out the monomial c^2, the expression $(ab + 1)^2 + 3(a + \lambda)^2 b^4 c^2$ restricts to the constant 1 on the fiber.

To resolve the function on the first canonical coordinate chart, we first pull back along the invertible linear map $(a, b, c) \mapsto (a, b, c - a)$ to obtain:

$$b^2 \left(c^2 + 3(a + \lambda)^2 b^2 \right)$$

Next, we blow up at $b = c = 0$, and pull back to the two canonical coordinate charts:

$$b^4 \left(c^2 + 3(a + \lambda)^2 \right) \qquad \text{and} \qquad b^2 c^2 \left(1 + 3(a + \lambda)^2 b^2 \right)$$

The second of these has normal crossings at any point (not just in the relevant fiber).

Case I. If $\lambda \neq 0$, we argue that the first of the previous two expressions has normal crossings on the fiber over $(\lambda, 0, 0)$ in the original space. The map from the first canonical coordinate chart to the original space is given by $(a, b, c) \mapsto (a + \lambda, b, b^2 c - ba - \lambda b)$ so the relevant fiber is $\{a = b = 0\}$. Factoring out the monomial b^4, the expression $c^2 + 3(a + \lambda)^2$ restricts to $c^2 + 3\lambda^2 > 0$ on the fiber. We introduce subscripts and summarize the maps from the three open sets resolving to normal crossings in notation similar to that appearing in [Wat09],

$$\begin{aligned} a &= a_1 + \lambda & &= a_2 + \lambda & &= a_3 + \lambda \\ b &= b_1 c_1 & &= b_2 & &= b_3 c_3 \\ c &= c_1 - \lambda b_1 c_1 & &= b_2^2 c_2 - a_2 b_2 - \lambda b_2 & &= b_3 c_3^2 - a_3 b_3 c_3 - \lambda b_3 c_3 \end{aligned}$$

The Jacobian determinants are c_1, b_2^2, and $b_3 c_3^2$, while the factored monomials are c_1^2, b_2^4, and $b_3^2 c_3^4$. We conclude that the real log canonical threshold of the original function K along the nonzero a-axis is $3/4$.

Case II. It remains to consider the case $\lambda = 0$, wherein the function $b^4 \left(c^2 + 3a^2 \right)$ does not have normal crossings. Blowing up along $a = c = 0$, the pull backs to the two canonical coordinate charts are given by:

$$a^2 b^4 \left(c^2 + 3 \right) \qquad \text{and} \qquad b^4 c^2 \left(1 + 3a^2 \right)$$

These have normal crossings. To summarize the $\lambda = 0$ case:

$$\begin{aligned} a &= a_1 & &= a_2 & &= a_3 & &= a_4c_4 \\ b &= b_1c_1 & &= b_2c_2 & &= b_3 & &= b_4 \\ c &= c_1 & &= b_2c_2^2 - a_2b_2c_2 & &= a_3b_3^2c_3 - a_3b_3 & &= b_4^2c_4 - a_4b_4c_4 \end{aligned}$$

The Jacobian determinants are $c_1, b_2c_2^2, a_3b_3^2, b_4^2c_4$. The factored monomials are c_1^2, $b_2^2c_2^4$, $a_3b_3^4$, and $b_4^4c_4^2$. We conclude that the real log canonical threshold of the original function K at the origin is $3/4$, the same as on the rest of the a-axis.

4.4.2. *Along the b-axis.* To produce normal crossings at $(0, \lambda, 0)$, we first pull back K along the translation $(a, b, c) \mapsto (a, b + \lambda, c)$ so that the point of interest is the origin:

$$(ab + \lambda a + c)^2 + 3a^2(b + \lambda)^4$$

Next, we pull back along the invertible linear map $(a, b, c) \mapsto (a, b, c - \lambda a)$ to obtain:

$$(ab + c)^2 + 3a^2(b + \lambda)^4$$

We then blow up along $a = c = 0$, and pull back to the two canonical coordinate charts:

$$a^2\left((b + c)^2 + 3(b + \lambda)^4\right) \qquad \text{and} \qquad c^2\left((ab + 1)^2 + 3a^2(b + \lambda)^4\right)$$

The second of these is the pullback of K under the map $(a, b, c) \mapsto (ac, b + \lambda, c - \lambda ac)$. The fiber of this map over $(0, \lambda, 0)$ is $\{b = c = 0\}$. Factoring out the monomial c^2, the expression $(ab + 1)^2 + 3a^2(b + \lambda)^4$ restricts to the constant $1 + 3a^2\lambda^4 > 0$ on the fiber, implying normal crossings.

Case I. If $\lambda \neq 0$, we claim that $a^2\left((b + c)^2 + 3(b + \lambda)^4\right)$ is also normal crossings. To see this, observe that this function is the pullback of K under $(a, b, c) \mapsto (a, b + \lambda, ac - \lambda a)$. The fiber of this map over $(0, \lambda, 0)$ is $\{a = b = 0\}$. Factoring out the monomial a^2, the expression $(b + c)^2 + 3(b + \lambda)^4$ restricts to $c^2 + 3\lambda^4$ on the fiber. As $\lambda \neq 0$, this implies that we have pulled back to normal crossings. To connect with notation appearing in [Wat09], we introduce subscripts and summarize the maps from the two open sets resolving to normal crossings as:

$$\begin{aligned} a &= a_1 & &= a_2c_2 \\ b &= b_1 + \lambda & &= b_2 + \lambda \\ c &= a_1c_1 - \lambda a_1 & &= c_2 - \lambda a_2c_2 \end{aligned}$$

The Jacobians are a_1 and c_2, while the factored monomials are a_1^2 and c_2^2. The real log canonical threshold of the original function K along the nonzero b-axis is hence 1.

Case II. Although we have already concluded that the real log canonical threshold of K at the origin is $3/4$, we can see this in a different way by continuing the procedure from above. When $\lambda = 0$, we do not have normal crossings for:

$$a^2\left((b + c)^2 + 3b^4\right)$$

We apply the origin-preserving diffeomorphism $(a, b, c) \mapsto (a, b, c - b)$ to obtain $a^2\left(c^2 + 3b^4\right)$. We then blow up along $b = c = 0$ to obtain the two pullbacks:

$$a^2 b^2 \left(c^2 + 3b^2\right) \qquad \text{and} \qquad a^2 c^2 \left(1 + 3b^2\right)$$

The second of these is normal crossings, while for the first we blow up again along $b = c = 0$ and obtain:

$$a^2 b^4 \left(c^2 + 3\right) \qquad \text{and} \qquad a^2 b^2 c^2 \left(1 + 3b^2\right)$$

The resulting maps can be summarized as:

$$\begin{aligned}
a &= a_1 c_1 && = a_2 && = a_3 && = a_4 \\
b &= b_1 && = b_2 c_2 && = b_3 && = b_4 c_4 \\
c &= c_1 && = a_2(b_2 - 1)c_2 && = a_3 b_3 (b_3 c_3 - 1) && = a_4 b_4 c_4 (c_4 - 1)
\end{aligned}$$

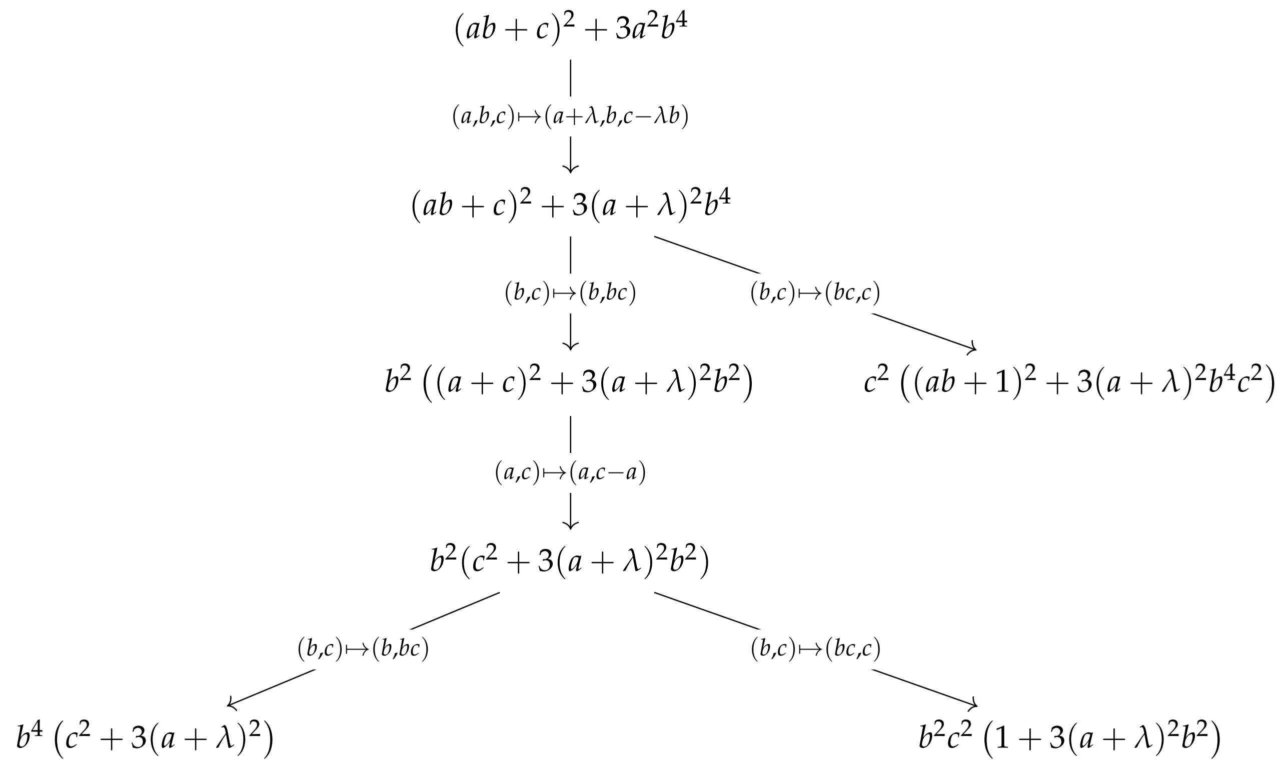
These are exactly the maps appearing in [Wat09, Example 3.18]. Keeping track of factored monomials and Jacobian determinants, one recovers a real log canonical threshold of $3/4$ for the origin.

4.4.3. *Summary.* We have:

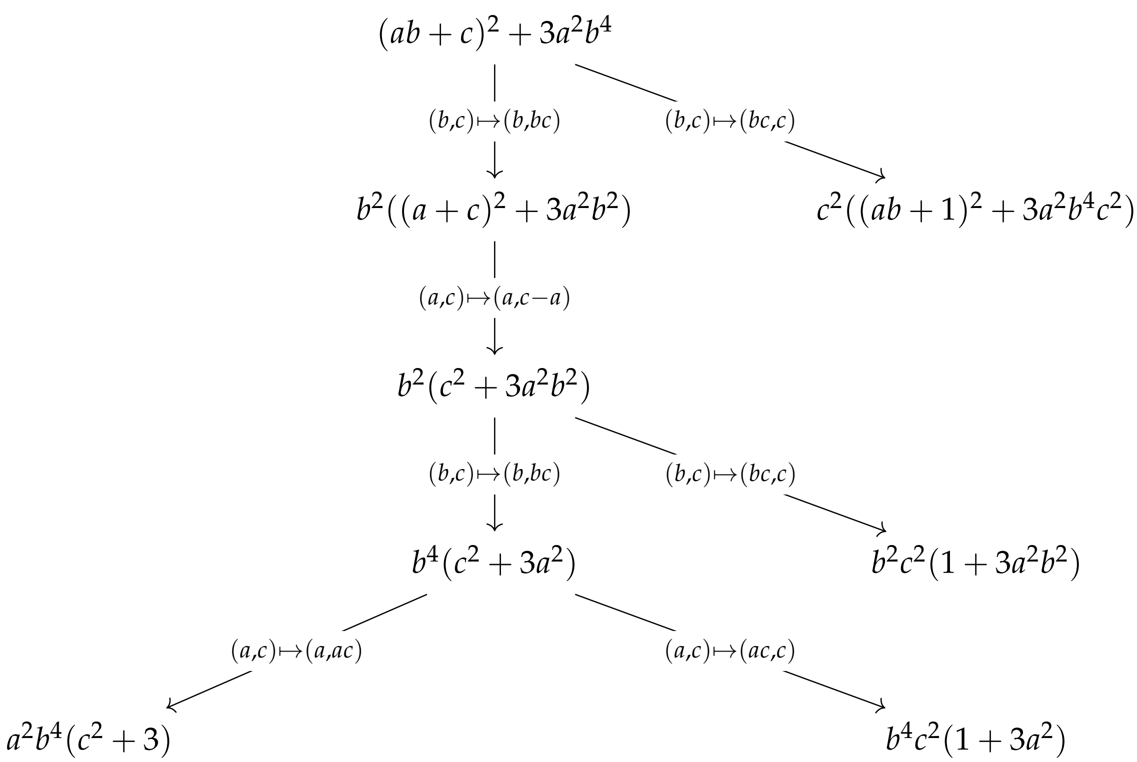
$$\mathrm{RLCT}(K, (a, b, 0)) = \begin{cases} 3/4 & \text{if } b = 0 \\ 1 & \text{if } a = 0 \text{ and } b \neq 0 \end{cases}$$

In particular, the real log canonical threshold at the origin is $3/4$. The (global) real log canonical threshold of $K^{-1}(0)$ is the minimum of these, namely $3/4$.

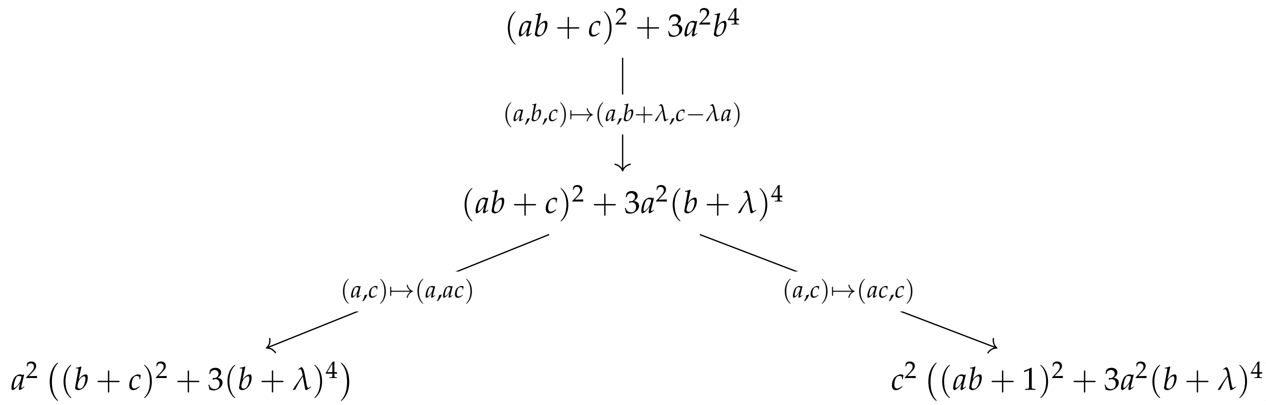
4.4.4. *Trees.* Along the nonzero a-axis, the point of interest is $(\lambda, 0, 0)$ for $\lambda \neq 0$.



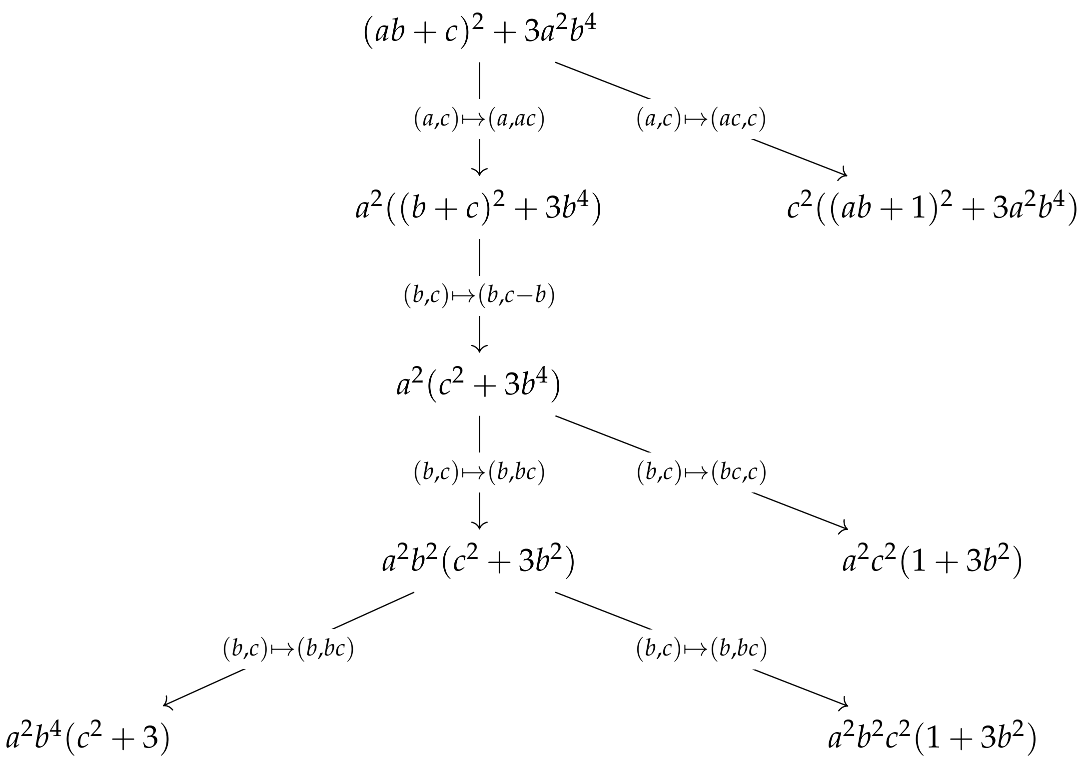
At the origin, approaching along the a-axis, we have:



Along the nonzero b-axis, the point of interest is $(0, \lambda, 0)$ for $\lambda \neq 0$.



At the origin, approaching along the b-axis, we have:



5. Limits

In this section, we give a different, equivalent, definition of flatness. Let f be a real analytic function defined on a neighborhood of the origin in $\mathbb{R}^d$, and suppose $f(0) = 0$. For any ϵ, consider the set of points in the unit square of $\mathbb{R}^d$ whose value under f is at most ϵ in absolute value, that is:

$$f^{-1}\left([-\epsilon, \epsilon]\right) \cap [-1, 1]^d$$

This is a closed neighborhood of the origin in $\mathbb{R}^n$. We set:

- S_ϵ to be the connected component of $f^{-1}\left([-\epsilon, \epsilon]\right) \cap [-1, 1]^d$ containing the origin.

- $V_f(\epsilon) := \mathrm{Vol}\,(S_\epsilon)$ to be the volume of S_ϵ.

Finally, we define the flatness of f at the origin as:

$$\text{Flatness}\,(f,0) = \frac{1}{\ln\left(\lim_{\epsilon\to 0}\frac{V_f(e\epsilon)}{V_f(\epsilon)}\right)} \tag{5.1}$$

where e is the base of the natural logarithm. Hence, the real log canonical threshold is the inverse of this expression:

$$\mathrm{RLCT}(f,0) = \ln\left(\lim_{\epsilon\to 0}\frac{V_f(e\epsilon)}{V_f(\epsilon)}\right)$$

A precise explanation of the equivalence between the two definitions of flatness is beyond the scope of these notes. Briefly, starting with a resolution $g : \mathcal{M} \to W$ as in Theorem 3.1, one chooses an atlas $\{(V_\alpha, \psi_\alpha)\}$ on $\mathcal{M}$ of coordinate charts satisfying the conditions of the theorem, and a partition of unity p_α relative to that atlas. Then

$$V_f(\epsilon) = \int_{S_f(\epsilon)} 1dx = \sum_\alpha \int_{V_\alpha\cap(g\circ\psi_\alpha)^{-1}(S_f(\epsilon))} p_\alpha(v)\det(\mathrm{Jac}_v(g\circ\psi_\alpha))dv$$

One uses the normal crossings property of $f\circ g\circ\psi_\alpha$ and $\det(\mathrm{Jac}_v(g\circ\psi_\alpha))$ to compute this integral, being careful with the domain of integration. Rather than delving into integration on manifolds, we confirm the equivalence in several easy examples.

Example 5.1. Let $f(x) = x^k$. Then $V_f(\epsilon) = 2\epsilon^{1/k}$, so that

$$\ln\left(\lim_{\epsilon\to 0}\frac{V_f(e\epsilon)}{V_f(\epsilon)}\right) = \ln\left(\lim_{\epsilon\to 0}\frac{2e^{1/k}\epsilon^{1/k}}{2\epsilon^{1/k}}\right) = \ln(e^{1/k}) = 1/k.$$

We confirm that the flatness is k.

Example 5.2. Consider the case of a monomial in two variables: $f(x,y) = x_1^{k_1}x_2^{k_2}$. Without loss of generality, suppose $k_2 \geq k_1$. If $k_2 > k_1$, then the x-axis direction away from the origin is "flatter" than the y-axis direction from the origin. More precisely, plotting the region $S_\epsilon \subseteq [-1,1]^2$, we see that the width along the x-axis is at least ϵ^{1/k_2}, while the width along the y-axis is at least ϵ^{1/k_1}. One can easily compute that the area of this region is:

$$V_f(\epsilon) = \frac{k_2}{k_2-k_1}\epsilon^{1/k_2} + \frac{k_1}{k_1-k_2}\epsilon^{1/k_1}$$

The leading exponent is $1/k_2$. On the other hand, if $k_1 = k_2 = k$, then

$$V_f(\epsilon) = \frac{k-\ln(\epsilon)}{k}\epsilon^{1/k}$$

which is the limit of the previous expression as $k_1 \to k_2$. In all cases, we obtain a flatness of $\max(k_1,k_2)$.

Example 5.3. Let $f(x) = x_1^{k_1} \cdots x_n^{k_d}$. If the k_i are all distinct, then:

$$V_f(\epsilon) = \sum_{i=1}^{d} \left(\prod_{j=1, j \neq i}^{d} \frac{k_i}{k_i - k_j} \right) \epsilon^{1/k_i}$$

Taking $\epsilon < 1$, the leading term is $\max_i(\epsilon^{1/k_i})$. Calculating the limit in Equation 5.1, one arrives at a flatness of $\max_i(k_i)$. The case of non-distinct k_i is more intricate but leads to the same result.

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Appendix A. Successive blowups

A.1. **Blowup of points on a manifold.** Let $\mathcal{M}$ be a d-dimensional real analytic manifold[4]. A coordinate chart at $p \in \mathcal{M}$ is a pair (V, ψ) where V is an open neighborhood of the origin in $\mathbb{R}^d$ and ψ is an injective real analytic map $V \hookrightarrow \mathcal{M}$ with $\psi(0) = p$. Let $W = \psi(V) \subseteq \mathcal{M}$ be the image of ψ, and let $\psi^\dagger : W \to \mathbb{R}^d$ be the left inverse of ψ.

Let $B_{\{0\}}(\mathbb{R}^d)$ be the blow up of the origin of $\mathbb{R}^d$. Given a coordinate chart (V, ψ) at $p \in \mathcal{M}$ as above, we can pull back along $\psi^\dagger$ to obtain the blow up of the neighborhood $W = \psi(V)$ at p:

$$B_p(W) := W \times_{\mathbb{R}^d} B_{\{0\}}(\mathbb{R}^d)$$

The projection to W has fiber $\mathbb{P}^{d-1}$ over p and is an analytic diffeomorphism on $W \setminus \{p\}$; hence we have a map $W \setminus \{p\} \hookrightarrow B_p(W)$. The blow up of p in $\mathcal{M}$ relative to W is defined as the coproduct (pushout) formed by glueing $\mathcal{M} \setminus \{p\}$ and $B_p(W)$ along $U \setminus \{p\}$:

$$B_{p,W}(\mathcal{M}) := (\mathcal{M} \setminus \{p\}) \coprod_{W \setminus \{p\}} B_p(W)$$

There is a map to M given by the obvious inclusion in the first cofactor and the projection to W in the second. The fiber of this map over p is a copy of $\mathbb{P}^{d-1}$, and the map is an analytic diffeomorphism on $\mathcal{M} \setminus \{p\}$. Up to analytic diffeomorphism, the blow up $B_{p,W}(\mathcal{M})$ does not depend on the choice of coordinate chart W, so we write simply $B_p(\mathcal{M})$. For distinct points $p, q \in \mathcal{M}$, blowups commute:

$$B_q(B_p(\mathcal{M})) = B_p(B_q(\mathcal{M}))$$

A.2. **Successive blowups.** A more interesting procedure is to blow up points in the fiber over p of the blowup. We describe this procedure in the case of blowing up the origin in $\mathbb{R}^d$, where the blow up admits canonical coordinate charts (U_i, ψ_i), for $i = 1, \ldots, d$, with the inverse maps $\phi_i : \mathbb{R}^d \to U_i$. Since ϕ_i is a diffeomorphism, we have a diffeomorphism of blowups:

$$\widetilde{\phi}_i : B_0(\mathbb{R}^d) \xrightarrow{\sim} B_{p_i}(U_i)$$

where $p_i := \psi_i(0) \in U_i$ is the image of the origin under ψ_i. Let U_{ij} denote the image of U_j under this map; these form an open cover of $B_{p_i}(U_i)$. Now, the point $p_i \notin U_j$ for $j \neq i$, so the blow up $B_{p_i}(B_0(\mathbb{R}^d))$ is covered by the $2d - 1$ coordinate charts:

$$U_1, \ldots, U_{i-1}, U_{i,1}, \ldots, U_{i,d}, U_{i+1}, \ldots, U_d$$

Let $p_{ij} \in U_{ij}$ be the center of the coordinate chart U_{ij}. We can then blowup again at p_{ij} to from $B_{p_{ij}}(B_{p_i}(B_p(\mathbb{R}^d)))$. Abstracting this observation, one can assign a blow up to any d-nary tree. See Figure 1 for an illustration in the case $d = 2$. Finally, one can ask whether this procedure generalizes to blowing up a general manifold; what fails is the existence of canonical coordinate charts U_i on the blow up $B_p(\mathcal{M})$, or, equivalently, a canonical coordinate chart W about p.

[4]In these notes we are particularly interested in real analytic manifolds, but many of the results carry over to smooth manifolds and algebraic varieties.

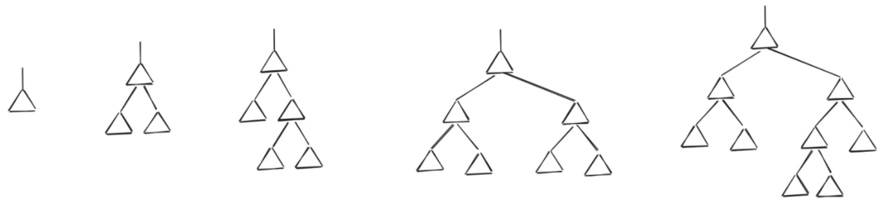

FIGURE 1. Binary trees correspond to successive blowups of $\mathbb{R}^2$. The trees above correspond to, in order, to the following blowups:

$$\mathbb{R}^2 \quad B_0(\mathbb{R}^2) \quad B_{p_2}(B_0(\mathbb{R}^2)) \quad B_{p_1}(B_{p_2}(\mathbb{R}^2)) \quad B_{p_1}(B_{p_{21}}(B_{p_2}(B_0(\mathbb{R}^2))))$$

We note that the fourth can also be written as $B_{p_2}(B_{p_1}(\mathbb{R}^2))$, while the fifth can be written in two addition ways: $B_{p_{21}}(B_{p_1}(B_{p_2}(B_0(\mathbb{R}^2))))$ and $B_{p_{21}}(B_{p_2}(B_{p_1}(B_0(\mathbb{R}^2))))$.

A.3. **Successive blow up equations.** We can write the successive blow up with equations. Specifically, $B_{0\in U_i}(B_0(\mathbb{R}^d))$ is the set of $(x, \ell, p) \in \mathbb{R}^d \times \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$ satisfying the following equations:

$$\begin{aligned} x_k\ell_j &= x_j\ell_k && \text{for all } j, k \\ \ell_k p_j &= \ell_j p_k && \text{for } j, k \neq i \\ x_i\ell_i p_j &= \ell_j p_i && \text{for } j \neq i \end{aligned}$$

We have a canonical open set $U_j = \{\ell_j \neq 0\}$ for $j \neq i$. In this open set, we can take $\ell_j = 1$, so that is is given by the set of (x, ℓ, p) such that:

$$\begin{aligned} x_k &= x_j\ell_k && \text{for } k \neq j \\ p_k &= \ell_k p_j && \text{for } k \neq i \\ p_i &= \ell_i^2 x_j p_j \end{aligned}$$

We see that p_j cannot be zero (which would lead to the contradiction that all the other homogenous coordinates of p are zero), so we can take $p_j = 1$. Then x_j, and ℓ_k for $k \neq j$ are free variables and determine the others. For example, $x_k = x_j\ell_k$ for $k \neq j$ (including the case $k = i$). We see that U_j is a copy of $\mathbb{R}^d$. There is also a canonical open set $U_{ij} = \{\ell_i \neq 0, p_j \neq 0\}$ for $j = 1, \ldots, d$. In this open set, we can take $\ell_i = 1$ and $p_j = 1$, so that is is given by the set of (x, ℓ, p) such that:

$$\begin{aligned} x_k &= \ell_k x_i && \text{for } k \neq i \\ \ell_k &= \ell_j p_k && \text{for } k \neq i, j \\ x_i &= \ell_j p_i \end{aligned}$$

The variable ℓ_j is free, as is p_k for $k \neq j$. These determine the other variables. For example,

$$x_k = \begin{cases} \ell_j p_i & \text{if } k = i \\ \ell_j^2 p_i & \text{if } k = j \\ (\ell_j)^2 p_k p_i & \text{otherwise} \end{cases}$$

$$U_j = \{\ell_j \neq 0\} = \{(x, \ell, p) \mid x_k = \frac{\ell_k}{\ell_j} x_j \ , \ p_k = \frac{\ell_k}{\ell_j}, \ , \text{for } k \neq j\}$$

for $j \neq i$, and $U_{ij} = \{\ell_i \neq 0, p_j \neq 0\}$ for $j = 1, \ldots, d$.

A.4. **Monoid.** Consider the monoid $\mathrm{SL}_2(\mathbb{N})$ of two by two matrices with non-negative integer entries and determinant 1.

Lemma A.1. *The monoid* $\mathrm{SL}_2(\mathbb{N})$ *is freely generated by the elements*

$$m_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad m_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Sketch of proof. Let M be the submonoid generated by m_1 and m_2. Consider the map $\max : \mathrm{SL}_2(\mathbb{N}) \to \mathbb{N}$ that picks out the largest entry. We argue by induction that $\max^{-1}(n) \subseteq M$ for any $n \in \mathbb{N}$. The base case is $n = 1$. Then one easily verifies that $\max^{-1}(1) = \{\mathrm{id}, m_1, m_2\} \subseteq M$. For the induction step, suppose we have shown that $\max^{-1}(n) \subseteq M$ for all $n < N$. Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{N})$ with $\max(a, b, c, d) = N$.

With these considerations in mind, we examine cases:

- It is impossible to have $a = 0$ while having the determinant equal to one.
- If $b = 0$, then $ad = 1$, so that $a = d = 1$. Then $g = m_1^c \in M$.
- If $0 < a \leq b$, then $c < d$ (this is because $1 = ad - bc \leq b(d - c)$). Then:

$$g = \begin{bmatrix} a & b - a \\ c & d - c \end{bmatrix} m_2$$

- If $0 < b < a$, then $c \geq d$ (this is because $1 = ad - bc > b(d - c) > d - c$). Either way, we have:

$$g = \begin{bmatrix} a - b & b \\ c - d & d \end{bmatrix} m_1$$

In the last two cases, the induction hypothesis implies that g is in M. This shows that $\mathrm{SL}_2(\mathbb{N})$ is generated by m_1 and m_2. By inspection of the cases above, we see that every element $g \in \mathrm{SL}_2(\mathbb{N})$ uniquely factors as $g = hm_1$ or $g = hm_2$. This implies that m_1 and m_2 freely generate the monoid. ☐

Hence, we can regard the elements of $\mathrm{SL}_2(\mathbb{N})$ as in bijection with the nodes of an infinite binary tree. To connect with blowups:

Proposition A.2. *Given* $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{N})$, *there is a sequence of successive blowups of* $\mathbb{R}^2$ *composing to a map* $\mathcal{M} \to \mathbb{R}^2$, *and a chart* $\mathbb{R}^n \overset{\psi}{\hookrightarrow} \mathcal{M}$ *such that* $\mu \circ \psi(x, y) = (x^a y^b, x^c y^d)$.

Sketch of proof. We proceed by induction on the length n of g as a word in m_1 and m_2. If $n = 0$, then g is the identity matrix, we take $\mu = \mathrm{id}_{\mathbb{R}^2}$. Otherwise, suppose $n > 1$. Then one of the following cases hold:

$$g = \begin{bmatrix} a-b & b \\ c-d & d \end{bmatrix} m_1 \qquad \text{or} \qquad g = \begin{bmatrix} a & b-a \\ c & d-c \end{bmatrix} m_2$$

Let's suppose the first case holds. By the induction hypothesis, there is a sequence of successive blowups of $\mathbb{R}^2$ composing to a map $\mathcal{M} \to \mathbb{R}^2$, and a chart $\mathbb{R}^n \overset{\psi}{\hookrightarrow} \mathcal{M}$ such that $\mu \circ \psi(x, y) = (x^{a-b}y^b, x^{c-d}y^d)$. Blowing up this chart at the origin and taking the first canonical chart yields the result. The argument for the second case is similar; one takes the relevant chart at the origin but takes the second canonical coordinate chart instead. □

Example A.3. Consider $f(x, y) = x^n + y^m$ with n, m relatively prime, and greater than 1. Let u, v be positive integers such that $nu - mv = 1$. These can be chosen uniquely satisfying:

$$0 < u < m \qquad \text{and} \qquad 0 < v < n$$

Set $u' = m - u$ and $v' = n - v$, so that u' and v' are the unique positive integers satisfying $mv' + nu' = 1$ and

$$0 < u' < m \qquad \text{and} \qquad 0 < v' < n$$

A short calculation shows that the matrix $g := \begin{bmatrix} u & u' \\ v & v' \end{bmatrix}$ belongs to $\mathrm{SL}_2(\mathbb{N})$. Writing this as a word in m_1 and m_2, we obtain a minimal sequence of successive blowups of $\mathbb{R}^2$ composing to a map $\mathcal{M} \to \mathbb{R}^2$, and a chart $\mathbb{R}^n \overset{\psi}{\hookrightarrow} \mathcal{M}$ such that $\mu \circ \psi(x, y) = (x^u y^{u'}, x^v y^{v'})$. Blowing up this chart at the origin we replace it by two additional charts with have:

$$\mu \circ \psi \circ \psi_1(x, y) = (x^m y^{u'}, x^n y^{v'}) \qquad \text{and} \qquad \mu \circ \psi \circ \psi_1(x, y) = (x^u y^m, x^v y^n)$$

Since all exponents are positive, the fiber over the origin is $\{xy = 0\}$ in both cases. The pullbacks of f to these charts are:

$$x^{mn}y^{nu'} + x^{mn}y^{mv'} = x^{mn}y^{nu'}(1 + y) \qquad \text{and} \qquad x^{nu}y^{mn} + x^{mv}y^{mn} = x^{mv}y^{mn}(x + 1)$$

One uses an argument similar to that appearing in Example 4.3 to show that these are normal crossings.

Additionally, one can show that the pullback of f to any other coordinate chart on $\mathcal{M}$ has normal crossings. This follows from a general fact, which we now state. Consider the function $x^{n_1}y^{m_1} + x^{n_2}y^{m_2}$, where the n_i and m_i are non-negative integers. Blowing up at the origin, the pullback of this function to at least one of the canonical open sets will have normal crossings. The verification of this fact is a straightforward case analysis.

Examining factored monomials and Jacobian determinants, one computes that the flatness of $x^n + y^m$ at the origin is $\frac{nm}{n+m}$. The real log canonical threshold is $\frac{1}{n} + \frac{1}{m} = \frac{n+m}{nm}$. The methods of this example can be used to show that the real log canonical threshold of the a sum $f(x) + g(y)$ of analytic functions is the sum of the real log canonical thresholds: $\mathrm{RLCT}(f(x) + g(y), (x^*, y^*)) = \mathrm{RLCT}(f, x^*) + \mathrm{RLCT}(g, y^*)$. In fact, it is a general result that the real log canonical threshold is additive over sums of disjoint variables.